Designing For Deep Mathematical Understanding

Mathematical Inquiry

Reasoning & Proof: (Chocolate Fix[i]; Consecutive Sums, Toothpick Problem)

- Develops mathematical conjectures; e.g. notes patterns and attempts to explain *why* they are true
- Attempts to generalize; i.e. systematically tests examples and counter-examples; considers under what conditions a statement or method is valid, possibly by testing extreme and/or special cases[ii]
- Distinguishes between necessary, possible, and impossible statements and identifies implications of what is known (e.g. "If 64 divides evenly by 8, then so must 128.")

Problem Solving: (Lamp Problem, Sharing Candy)

- Develops a plan, modifies it as needed, simplifies if possible;
- Identifies sub-problems and relates them to each other and to the main problem;
- Considers strengths and weaknesses of various strategies
- Considers similarities and differences between different strategies, different problems, and variations on a particular problem

Modeling / Mathematizing[iii]: (Footprint Problem, Ice Melt Problem)

- Describes situations mathematically by selecting relevant aspects of a situation and describing them with some form of mathematical symbol system (or model)
- Considers strengths / weaknesses of model (e.g. "Is weight ÷ track area an appropriate way to describe 'sinkability'?");
- Generalizes models of individual situations to models that work in a variety of situations
- Compares models to see what can or cannot be represented and to see what insights can emerge from attempting to do so.

Developing and Using Mathematical Procedures[iv]:

- Distinguishes between aspects of procedure that are necessary and those that are arbitrary (i.e. agreed-upon conventions) [v]
- Compares effectiveness of invented strategies with conventional procedures (there is often a trade-off between transparency and efficiency)
- Uses efficient procedures appropriately and accurately (note contrast with *mathematizing* above).
- Considers reasonableness of answers

Extending and Connecting Concepts:

- Understands connections between various mathematical topics and representations (e.g. connections between multiplication and division; linear relations and proportionality)
- Systematically explores variations of a particular concept; considers which dimensions might vary and to what degree[vi]
- Systematically identifies all possible cases that fit particular constraints; where appropriate, shows that no others are possible (e.g. How many ways can you make a $0.50 with nickels, dimes, and/or quarters?)[vii]
- Attends to and resolves discrepancies between aspects of understanding[viii]

***Mathematical* Work Habits** (Productive Disposition)

- Considers alternative ideas
- Tolerates ambiguity
- Willing to try own ideas before seeking help
- Engages in a state of flow, characterized by extended periods of deep thinking
- Experiences "aha!" moments that are often characterized by the excitement of trying to communicate ideas to the teacher or other students; e.g. involving loud expressions accompanied by bodily movement
- Appreciates elegance; i.e. the appeal of simple but powerful arguments that help with solving problems or understanding mathematical concepts

Establishing and Supporting *Mathematical* Community

- Contributes to class discussion re: the development of ideas and solving of problems
- Connects contributions to what others have said or done (This goes with....; I agree with....; I disagree with...; I think I see what ... means by ...; Another way of saying that might be….)
- Respects other people and ideas; i.e. works hard to understand other views (asks questions, paraphrases, etc.); develops clear arguments to convince others of own views

Communication

- Shows work;
- Selects or develops appropriate representations to explain and develop key ideas (uses writing, charts, diagrams, models, etc.)
- Organizes complex ideas
- Uses appropriate mathematical terminology and notation to express their ideas
- Uses precise, concise, and unambiguous language to describe mathematical objects to a desired level of specificity (e.g. "a quadrilateral with 2 sets of parallel lines" might be a rectangle, square, rhombus, or parallelogram; parallelogram includes all of these; if other features are important, a more specific word could be chosen)

Strong Work in Mathematics

Mathematical Inquiry: Are the students engaged in (a) *mathematizing*; (b) *developing, refining, and comparing solution strategies*; and/or (c) *making and testing mathematical conjectures*? When developing and comparing procedures, do students recognize a trade-off between *efficiency* and *transparency*? Do they distinguish between rules and / or terms that are *arbitrary* (merely conventional) and those that are defined by *mathematical necessity*? Do they notice *connections* between various topics they are studying? Do they systematically explore potential variations? Do they work to resolve discrepancies in their understanding?
Evidence
Mathematical Work Habits: Are students confidently voicing ideas that are partially formed or that may turn out to be wrong? Are they persistent in testing their ideas?
Evidence
Mathematical Community: Are students *contributing* their own ideas, *respecting* other's contributions, and attempting to *connect* various contributions?
Evidence
Communication: Are students seeking ways to *show* and *organize* their ideas? Are they attempting to express their ideas with *mathematical terminology* that is shared by the broader mathematical community?
Evidence

A few notes of clarification….

Mathematizing is much different from merely applying teacher-given procedures to solve a mathematical problem. When students mathematize, they describe a situation in ways that make particular relationships more apparent. They might draw a graph, write an algebraic expression, assign categories, etc., but they do so *to make sense of something*, not just to practice using a pre-determined algorithm, formula, or procedure.

For example, many students develop algebraic expressions to describe how they count toothpicks in the toothpick problem, but they don't do it simply because someone told them to "make an expression." They do so to express their method of counting more efficiently, and the expressions evolve as various students compare their solutions.

As students **develop, refine, and compare** their solution strategies, they develop a more connected sense of why those strategies work and what they can do.

Making and testing mathematical conjectures can take place any time students are involved in developing their own strategies and considering the limits of their application. Students can readily see that a rectangle cut in half diagonally produces a triangle whose area can be described by $\frac{1}{2}bh$. But does this apply to *all* triangles?

Mathematical variation can be **understood in terms of "dimensions of possible variation"** and "**range of permissible change"** (Watson & Mason, 2005; 2006). When variation is done systematically, richer understanding of concepts may be developed. For example, students might generate a whole collection of triangles with $b = 1$, $h = 1$. What does such a collection look like? What do they all have in common? How can they differ? What happens if $h = 2$?

The trade-off between efficiency and transparency is often apparent when students approach a new idea. Students' early attempts to develop procedures tend to be cumbersome. With a little feedback, though, they do work - and more importantly, they make sense. What is sometimes labeled "new math" is (rightly) criticized for leaving students with no efficient strategies. But it is important to consider how students' varied approaches are related to each other as well as to standard algorithms / procedures.

For example, the long division algorithm is efficient within certain bounds. But students don't usually make an intuitive extension to decimals, and it's not particularly helpful with fractions. *It's efficient so long as you stay within the bounds to which it applies, but (for most) it's not transparent in a way that makes extension / adaptation to new situations possible*. Students might adapt the methods below to work with decimals or fractions, if only because here they're still thinking about the fact that they're actually dividing, rather than "seeing if the 7 goes into 36," "bringing down the 5," etc.:

In dividing 365 / 7, one student did the following:
350 / 7 = 50
10 / 7 = 1 R3
5/7 = R5
Total = 51 R8 = 52 R1

Another did this:
50/7 = 7R1
Do this 7 times, and you get 49 R7, or 50
10/7 = 1 R3
5/7 = R5
Total = 50 + 1 R8 = 52 R1

Neither provides a foolproof method that would work in any whole-number situation, but students are building a deeper understanding of division as they work. Eventually, this deeper understanding can (and should be) be bridged with conventional and efficient procedures.

The **distinction between arbitrary and necessary** (Hewitt; 1999, 2001a, 2001b) *explicitly* recognizes that some knowledge is purely conventional (e.g. a superscript 3 means to the power of 3) and must be directly taught and memorized, while other knowledge can be figured out based on what is already known. For example, in $3^2 * 3^3 = 3^5$; does adding the exponents *always* work? Under what conditions? What happens with $3^2 / 3^3$? $(3^2)^3$ $3^2)^3)^4$? If teacher and students agree on the meaning of the exponent, the exponent laws follow as necessary consequences, and students can be involved in developing them (and perhaps coming up with some of their own).

Even when teachers recognize the distinction between arbitrary and necessary, students may not. They struggle with what they perceive as arbitrary. In what may seem like an extreme case, some students even memorize multiplication tables as though they were arbitrary and are surprised to discover that, say, “3x2” actually *means* something. Imagine trying to memorize that many math facts as though they were purely arbitrary! It would be like trying to memorize a random sequence of letters or words.

Mathematicians have long pointed to the aesthetics of mathematical arguments or ideas (*cf* Hadamard, 1945). Research in mathematics education also emphasizes the importance of the **affective domain** (e.g. attending to elegance, visual appeal, surprise) to learning and doing mathematics (*cf*. Sinclair, 2006). Further, all students are capable of and need opportunities to experience the intellectual enjoyment of the “aha!' moments of insight that can happen when they solve a problem or make an important connection between mathematical ideas (*cf*. Liljedahl, 2004). Related to this is what (Csíkszentmihályi, 1996) called the state of “flow,” in which people fully engage in a task for an extensive period of time. Current research in neuroscience has associated meaningful learning with both the state of flow and the aha! moment (discussed further in Friesen, 2007; OECD, 2007). As teachers, it can be tempting to guide too much, thereby removing opportunities for students to have their own ahas[ix].

With accessible tasks and sufficient time and space, all students are capable of this experience. Even noticing the pattern in the sums of 10+1, 10+2, 10+3, 10+4, etc. can be an aha if we don’t teach it as a “strategy” to help with the learning of addition facts. Many good problems have multiple entry points. *Chocolate Fix* involves many levels of deductive logic. Anybody who can count past 100 can come up with a strategy for the “Toothpick Problem.” In “Sharing Candy,” most middle school students can figure out the possibilities emerging from 2 kids sharing the candy with one candy left over. In “Consecutive Integers,” most can figure out and explain a pattern for 2 consecutive integers. The “Lamp Problem” typically evokes 2 conflicting responses, both accessible to anybody with very basic arithmetic; resolving the conflict is the main challenge.

A **mathematical community** must do more than provide opportunities for students to publicly share their ideas. It should be a place where those ideas interact and evolve together (Davis & Simmt, 2003). For this to happen, teachers must structure the sharing and students must develop strategies for questioning and responding to one another.

When students use **mathematical terminology** to express their ideas, it is important that they are first allowed to develop their ideas. The terms are merely applied to understandings they have already worked with extensively.

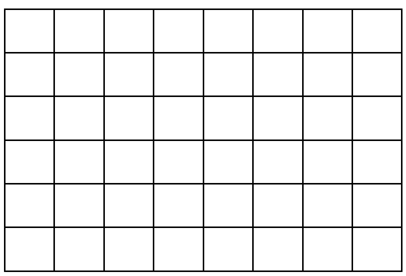
Sample Problems

While none of the categories of mathematical inquiry can stand alone, particular problems may have stronger emphasis on certain modes of inquiry. If engaged deeply, the ones offered here offer much to think about.

Chocolate Fix: A deductive logic puzzle by *ThinkFun®* available as a board game or as an iTunes app

Consecutive Sums: Which numbers can be expressed as the sum of consecutive numbers? In how many ways can a particular number be so written? (e.g. 21 can be written as 6+7+8, as 1+2+3+4+5+6, or as 10+11). (from Mason, Burton, & Stacey, 2010).

Toothpick Problem: How many toothpicks would it take to make the shape below (one side of each small square is made up of one toothpick)? Which methods show more efficient ways of counting? Can you find an efficient method that works for a rectangle of any size? (Adapted from Mason, Burton, & Stacey, 2010)



Lamp Problem: Suppose you buy an antique lamp for $7, then sell it for $8. You buy the same lamp back for $9, then sell it for $10. How much profit do you make? (adapted from Schultz, 1977/1982, p.12)

Sharing Candy: A bag of candy sits on a table. If two kids share all the candy so that each one gets the same number of pieces, there's one candy left over. If three kids share the same candy equally, there are two candies left over. If four kids share the candy equally, there are three candies left over. If five kids share the candy equally, there are four candies left over. If six kids share the candy equally, there are five candies left over. *How many candies are in the bag? Is there more than one possibility? What happens if you add the requirement that if seven kids share the candy equally, there are six candies left over? Eight with seven? Nine with eight? Can this be extended indefinitely? What happens if there are two instead of one left over each time?*

Footprint Problem: How accurately can you predict a person's height from a footprint?

Ice Melt Problem: Given a funnel of ice melting into a graduated cylinder for an unknown amount of time, can you figure out when the ice started melting? (Adapted from Wise, 1990)

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Notes

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[i] Sample problems are included in this document. While no problem fits neatly into one category of inquiry, some lend themselves particularly well to one type or another.

[ii] Mason, Burton, and Stacey (2010) developed an excellent resource for developing mathematical reasoning and problem solving. First published in 1982, it includes a large collection of rich problems. The most recent (2010) edition also considers how these might align with traditional curriculum distinctions.

[iii] Richard Lesh and his colleagues have done much to distinguish mathematical modeling from traditional problem solving (*cf.* Lesh & Harel, 2003). Dan Meyer's (2012) description of the "ladder of abstraction" is also helpful in clarifying the distinction between mathematizing and merely applying mathematical procedures. Catherine Fosnot and her colleagues have developed excellent descriptions of mathematizing at the K-6 level (Fosnot & Dolk; 2001a, 2001b, 2002).

[iv] The categories used here are influenced in part by Kilpatrick, Swafford, and Findell's (2001) five strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competency, adaptive reasoning, and productive disposition.

[v] Dave Hewitt (1999, 2001a, 2001b) did a wonderful job of drawing attention to the distinction between what is arbitrary and necessary in mathematics. The story of "Benny" (Erlwanger, 1973) provides a classic example of a student who treats mathematical procedure as arbitrary. An interesting discussion of this piece may be found at http://blog.mathed.net/2011/07/rysk-erlwangers-bennys-conception-of.html.

[vi] Anne Watson and John Mason (2006) talk about the "dimensions of possible variation" and "range of permissible change" afforded by particular mathematical concepts.

[vii]John Mighton provides an excellent collection of tasks that allow students to practice working systematically in Chapter 10 of *The Myth of Ability.*

[viii] Suzanne Donovan and John Bransford emphasized the importance of "engaging resilient preconceptions" in mathematics. They offered three strategies for doing so: (1) Draw on knowledge and experiences that students commonly bring to the classroom but are generally not activated with regard to the topic of study (p. 569); (2) Provide opportunities for students to experience discrepant events that allow them to come to terms with the shortcomings in their everyday models (p. 571); and (3) Provide students with narrative accounts of the discovery of (targeted) knowledge or the development of (targeted) tools (p. 573).

[ix] Jung-Beeman et al (2004) observed the brain activity of participants completing "Compound Remote Association Puzzles," in which they have to think of a word that could be combined with each of three prompts; e.g. pine / crab / sauce (*apple* allows pineapple, crabapple, and applesauce). The answer was associated with a rush of gamma waves in the right hemisphere. Simply reading the answers, however, doesn't feel very interesting. Try these: cottage / swiss / cake (cheese); iron / shovel / engine (steam); flake / mobile / cone (snow). Now try these (no answers given): cream / skate / water; dew / comb / bee; political / surprise / line.