LEARNING MATHEMATICS IN AN ACCESSIBLE CLASSROOM

Research Report

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LEARNING MATHEMATICS IN AN ACCESSIBLE CLASSROOM

Background Information And Project Description

"Math. The bane of my existence for as many years as I can count. I cannot relate it to my life or become interested in what I'm learning. I find it boring and cannot find any way to apply myself to it since I rarely understand it." (ATA, 2003, p.28)

"*Was ever a human activity preached so differently from how it was practiced, taught so clumsily, learned so grudgingly, its light buried beneath so many bushels, as mathematics*?" (Kaplan & Kaplan, 2007, p.117)

Introduction

Teaching And Learning Mathematics

Perhaps more than any other discipline, the teaching of mathematics lends itself to procedural recipes where students memorize and duplicate procedures by rote: if it looks like this, do that to it. "For many, mathematics is the torture of tests, homework and problems, problems, problems" (Burger and Starbird, 2005, p.xi).

"If one believes that mathematics is mostly a set of procedures—rules and truths—and the goal is to help students become proficient executors of the procedures, then it is understandable that mathematics would be learned best by mastering the material incrementally, piece by piece" (Stigler and Hiebert, 1999, p.90). Teaching practices that commonly flow from this view are demonstration, repetition and individual practice. In addition to being a misunderstanding of the discipline of mathematics itself, this belief also colors people's views about who can learn mathematics. Curricula and teaching practices are often based on what Mighton (2007, p.2) calls a destructive ignorance "that leads us, even in this affluent age, to neglect the majority of children by educating them in schools in which only a small minority are expected to naturally love or excel at learning", particularly mathematics. He insists that too many students lose faith in their own intelligence, and too much effort is directed at creating artificial differences between fast and slow, gifted and "special", advanced and delayed.

And worse yet, procedural approaches to the teaching of mathematics that create problems of understanding and engagement are applied with even more vigor in remedial programs designed to help those very students for whom such practices did not work in the first place (Swain and Swan, 2007).

Fuson, Kalchman and Bransford (2005), Mighton (2003, 2007), Stigler and Hiebert (1999) and Swain and Swan (2007) argue that other approaches are needed to help students learn mathematics. "Today, mathematics education faces two major challenges: raising the floor by expanding achievement for all, and lifting the ceiling of achievement to better prepare future leaders in mathematics, as well as in science, engineering, and technology. At first glance, these appear to be mutually exclusive" (Research Points, 2006, p.1). But are they? Is it possible to design learning that engages the vast majority of students in higher mathematics learning?

The purpose of this study is to determine whether the principles of Universal Design for Learning (UDL) result in increased student mathematical proficiency and achievement for all students in a typical Grade 7 classroom, including those with identified learning needs.

Kilpatrick, Swafford and Findell (2001) define mathematical proficiency in terms of five intertwining strands:

- *conceptual understanding* – an understanding of concepts, operations and relations. Conceptual understanding frequently results in students' comprehending connections and similarities among interrelated facts.
- *procedural fluency* – flexibility, accuracy and efficiency in implementing appropriate procedures. Skill in proficiency includes the knowledge of when and how to use procedures. This includes efficiency and accuracy in basic computations.
- *strategic competence* – the ability to formulate, represent and solve mathematical problems. This is similar to problem solving. Strategic competence, conceptual understanding and procedural fluency are mutually supportive.
- *adaptive reasoning* - the capacity to think logically about concepts and conceptual relationships. Reasoning is needed to navigate through the various procedures, facts and concepts required to arrive at solutions.
- *productive disposition* – positive perceptions about mathematics. Productive disposition develops as students gain more mathematical understanding and become capable of learning and doing mathematics.

These five intertwining strands map directly to the three principles of learning identified by Bransford, Brown and Cocking (2000):

- Engaging students' preconceptions and building on existing knowledge builds a productive disposition.
- A foundation of factual knowledge (procedural fluency) tied to a conceptual framework (conceptual understanding) and "organized in a way to facilitate retrieval and problem solving (strategic competence)"

(Fuson, Kalchman & Bransford, 2005, p.218) builds mathematical proficiency. "Because mathematics has traditionally been taught with an emphasis on procedure, adults who were taught this way may initially have difficulty identifying or using the core conceptual understandings in a mathematics domain" (Fuson, Kalchman & Bransford, 2005, p.233).

- Ongoing sense making or adaptive reasoning, reflection and problem solving or strategic competence support a metacognitive approach which enables student self-monitoring.

Guided by the definition of mathematical proficiency and principles of how people learn derived from the learning sciences, the research team[1] designed interventions in teaching concepts of geometry to a class of Grade 7 students of mixed ability as determined by Individual Program Plans (IPP's) in place at the time of the research. The design of the intervention was based on the following assumptions about effective mathematics instruction:

- connections between and among the proficiency strands are inherent;
- every student can make progress along every strand;
- mathematical reasoning as a set of practices and norms is (1) collective, not merely individual or idiosyncratic, and (2) rooted in the discipline of mathematics" (Ball & Bass, 2003, p.3);
- teaching students to reason mathematically requires them to formulate conjectures, create and use networks of concepts and processes, justify solutions and find proof that their solutions are valid;
- lessons are structured to address misconceptions, build coherence and help students make connections that are not inherently obvious; and
- effective instructional design integrates a sequence of instructional activities into a coherent whole (Fernandez, Yoshida, & Stigler, 1992). Students need coherent mental representations in order to use those representations to form new knowledge

In the past, such instructional design was generally reserved for students thought to be innately talented in mathematics. Our interest was to test the hypothesis that interventions in instructional design would permit all students in an ordinary classroom to build and demonstrate mathematical proficiencies as defined in this research report, including students on IPP's designed to address their previously identified learning challenges.

Universal Design For Learning

Three elements converged in decisions about the research intervention designed for this study: a clear definition of mathematical proficiency; alignment of that definition with findings from the learning sciences; and key assumptions about the teaching of mathematics. A fourth element was required in order to

[1] Our use of this term includes the teacher.

specifically address (1) students with identified learning needs and (2) the role of technology in the mathematics classroom.

"Recent educational innovations, such as differentiated instruction and universal design for learning (UDL), offer insights into proactively planning instruction that embraces academic diversity" characteristic of most ordinary classrooms (Edyburn, 2006a, p.21). UDL is grounded in emerging insights about brain development, learning, and digital media. Rose and Meyer (2002, 2006) and Rose, Meyer and Hitchcock (2005) observe that the disconnect between an increasingly diverse student population and a "one-size-fits-all" curriculum will not produce the desired academic achievement gains expected in the 21st century. Drawing on the historical application of universal design in architecture, they advance UDL as a means of focusing educational research, development, and practice on understanding diversity, technology, and learning.

This study derives from such a focus.

Because of the structure of our current education system, which makes sharp distinctions between "regular" and "coded" students, UDL has been taken up most seriously in special education where issues of access to high quality learning experiences for variously identified special needs students carry a particular urgency (Firchow 2002; Meo 2005). But all proponents of UDL actually make much larger claims for their ideas:

- that diversity in the classroom is the norm rather than a problem to be fixed;
- that paying attention to what does—and does not—work for students generally relegated to the margins will improve learning for all;
- that all students can meet similar learning goals if curricula, instruction and assessment are radically reconceptualized; and
- that effective use of technology enables all students to represent, express and engage with ideas in multiple ways not generally seen in conventional classrooms.

To the extent that principles of UDL are increasingly familiar in Alberta, we will not summarize the field in general. Rather, we draw on the following key principles as they provide a focus for dismantling procedural methods of teaching mathematics in favor of developing mathematical proficiency for all.

1. Procedural approaches to the teaching of mathematics privilege naked independence (Edyburn 2006a, p.22): the notion that completing tasks without performance-enhancing access to technology is superior to performance that is enhanced through technology. This out-dated formulation of what it means to be an educated person ensures that academic achievement is reserved only for able-bodied individuals, and only for those individuals who are able to succeed without external support, resources or technology. For many students, "technology can be

the difference between students with special needs sitting in a classroom watching others participate and all students participating fully" (Bausch and Hasselbring, 2005, p.9). And for all, access to a wide variety of digital media permits mathematical explorations that are difficult or impossible with only pencil and paper. These media include (but are not limited to) spreadsheets and databases, simulations, software such as Geometer's Sketchpad™, computer assisted design, programming, interactive games, etc. Accessible classrooms are media rich (Friesen 2006).

2. Disability can be conceived as a mismatch between the learner's needs and the education offered (Rothberg and Treviranus 2006). Rather than conceptualized as a personal trait, disability can be seen as an artifact of the way children are taught (Mighton, 2007, p.24). Many students come to dislike mathematics because of their experiences in school. Even worse, they lose faith in themselves as learners. That is, the mismatch between what is offered and what they need in order to become engaged, enthusiastic and proficient learners may actually create what come to be identified as mathematical learning disabilities.
3. UDL design principles focus on creating clear goals, flexible methods and materials, and embedded assessments that enable all learners including those with disabilities to access knowledge, participate and progress.
4. Learning is about deep understanding, constructing knowledge and developing skills and thus requires a careful balance of support, challenge and opportunity. "But the most fundamental change will come in our understanding of goals. The ultimate educational goals will no longer be about the mastery of content (content will be available everywhere, anytime, electronically) but about the mastery of learning" (Rose & Meyer, 2000a).
5. UDL calls for:
    - *Multiple means of representation,* to give learners various ways of acquiring information and knowledge,
    - *Multiple means of expression,* to provide learners alternatives for demonstrating what they know,
    - *Multiple means of engagement,* to tap into learners' interests, offer appropriate challenges, and increase motivation

(CAST, 2006)

Accessing The General Curriculum

In educational research, the word "curriculum" can be defined in both narrow and broad ways. In its narrowest sense, curriculum can be taken to mean the mandated Program of Studies, or core curricula.

For the purposes of this study, we will use the word curriculum in a broader sense to include:

- The mandated Program of Studies for mathematics in Alberta
- Instructional design of a four week study of geometry, which is identified in the Program of Studies under the topic of Shape and Space
- Instructional strategies and pedagogy
- Resources and supports
- Assessment

Conventionally, when students have difficulties learning concepts and skills outlined in the Program of Studies, modifications to instructional strategies, pedagogy and resources and support are provided. Generally, material thought to be too complex or difficult for them to master is broken down into smaller pieces. Often they are given work that appears easier, or that requires increased practice and skill-building drill, sometimes with the aid of classroom assistants and pull-out programs.

The prevailing assumption is that mathematical difficulties lie in the inherent inability of the individual student to master the mandated program of studies in ways that are unproblematic for normal or regular students. Accommodations are provided to remediate these difficulties to the extent possible for each disabled learner, however disability is defined for that individual. In a nutshell, differences among and between learners are generally regarded as a problem to be remedied with modification to existing programs.

UDL challenges this assumption in fundamental ways (Rose & Meyer 2002; Hendricks and Daley 2004):

- Students with disabilities are understood to fall along a continuum of learner differences among all students rather than constituting a separate category;
- Teacher adjustments for learner differences should occur for all students, not just those with disabilities;
- Curriculum materials should be varied and diverse for all learners, including digital and online resources;
- Instead of “fixing” students so that they can learn from a set curriculum, resources and teaching methods and designs for learning need to be flexible enough to accommodate a wide range of learner differences from the outset.

Too many students do not “get” conventional approaches to the teaching of mathematics. This is not limited to those with identified learning disabilities but also includes “non-coded” learners struggling with the culture, language and access to learning.

It also includes those who just simply come to dislike math.

Designing "Dynamic" Assessment

Fair and accurate assessments of learning allow all students to demonstrate their progress and understanding in multiple ways. Traditional print-based assessments often measure facts and recall, and conventional assessment tools such as multiple choice tests or fill-in-the blank worksheets often block a true picture of the learning (Rose, Meyer and Hitchcock, 2005).

Dynamic assessment provides learning scaffolds and feedback to the learner and to the educator. Multiple means of expression and assessment enable both educators and students to assess what they currently know and to identify and plan the required next steps. Dynamic assessment aligns closely with teaching goals and methods. Assessment *of*, *for* and *as* learning is an integral part of the instructional design process. It is also a key aspect of the design of this research study. Constant assessment of students' developing proficiencies and their misconceptions guided the direction of the intervention on an on-going basis. Thus, while conventional measures of achievement derived from tests following instruction are helpful in assessing the end results of teaching, they are only one of myriad forms of assessment designed to provide direct feedback to both learner and teacher on students' developing proficiency throughout a unit of study.

Researchers at Boston College found that when students were given access to computers to digitally compose their answers to written portions of tests, they scored significantly higher than those using paper and pencil (Dolan and Hall 2001). Digital tools and expressive media give students a wide range of opportunities to represent and express what they are learning. When such media have built-in capacities for interaction, they also provide immediate feedback on performance. Consider, for example, how engaging a computer game can be as players strive to develop the skills they need to move to the next level of play. The learning environment, itself, is designed to give precise feedback. No gamer needs to wait for a Friday morning quiz to see if she is progressing.

Inflexible standardized assessment that does not meet the learning needs of students confounds the measurement of knowledge and abilities (Dolan and Hall 2001), even when that assessment is computer-based. If assessment practices remain a one-size-fits-all method of sorting out ability hierarchies in the classroom, they will give incomplete pictures of the multiple ways in which individuals develop their proficiencies. Well-designed, embedded, dynamic assessment practices have the potential to remove many of the current barriers to learning in the mathematics classroom.

Assistive Technology (AT), Digital Media and Universal Design for Learning (UDL)

It is becoming increasing difficult to differentiate between assistive technologies, digital media and universal design for learning; therefore, it is not unusual that many teachers confuse assistive technologies and UDL. By definition, “AT is an intervention that is explored after a performance problem is identified. On the other hand, UDL is proactive instructional design that seeks to build learning environments and instructional materials with supports that enable all students to achieve the academic standards despite differences” (Edyburn, 2005). These definitions seem fairly straightforward; however, divisions between AT, digital media and UDL are actually less distinct.

In the mathematics classroom, calculators could be considered an assistive technology because they are "an item, piece of equipment, or product system which can be used to increase, maintain, or improve the functional capabilities[2]" of the student. In Alberta, “assistive technology for learning (ATL) is defined as the devices, media and services used by students with physical, sensory, cognitive, speech, learning or behavioural disabilities to actively engage in learning and to achieve their individual learning goals” (Alberta Education, 2006, ch.9, p.1). Web resources such as WebMath (http://www.webmath.com/) can also be considered an ATL. Edyburn (2006c) reports, “the performance data associated with [a] student using a technology intervention (i.e., WebMath) reveals a considerable difference in performance gains of 40 percent and 50 percent during the short period in which this intervention was provided” (p.4).

For some students, a lack of procedural fluency impedes their ability to gain mathematical proficiency. Therefore, an educational conundrum arises. Should the teacher provide students with calculators or WebMath only *after* a performance problem has been identified? And for how long must a student manifest this performance problem before the appropriate AT or ATL is provided? One year, two years, five years?

> Rothberg and Treviranus (2006) challenge us to expand our definition of a disability within a learning context. They contend that a disability is a mismatch between the learner’s needs and the education offered. They further suggest that a learning disability is not a personal trait but an artifact of relationship between the learning environment or education delivery. They argue for the need to create accessible classrooms, which involves the ability of the learning environment itself to adjust to the needs of all learners. (Friesen, 2006, p.7).

Working with the principles of UDL, the teacher anticipates that some students might have difficulties with procedural fluency and therefore proactively builds the use of calculators, WebMath or any other instructional materials into the

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[2] (Individuals with Disabilities Education ACT (IDEA) 20, USC, Ch 33, Section 1401 (25) US

instructional design.

In addition, the teacher would design multiple opportunities for all students to use a wide range of digital environments to represent, express and engage with mathematical ideas.

Setting The Alberta Context

Alberta students consistently score very well on international (PISA, TIMSS) and national (SAIP) mathematics studies[3]. Given such high international and national standings, many educators in Alberta might question why Alberta Education would be interested in ensuring even higher achievement for all students in the area of mathematics. Perhaps this can be best explained by a brief conversation that Dr. Friesen had with an individual from Alberta Education's Assessment Branch. In discussing Alberta's success on the recently released PISA 2006 findings, in which Alberta scored second only to Finland, this person stated, "We still have work to do. There is no place to stand still. If you are standing still you are actually going backwards."

This research study is designed to encourage continued conversation about going forward with mathematics education in this province, particularly in terms of:

- better meeting the needs of Alberta's increasingly diverse student population;
- reducing the number of students who give up on the study of mathematics;
- determining whether achievement scores alone give a finely grained enough view of actual mathematical proficiency.

Building On Previous Alberta Research Studies

In 2006, Alberta Education contracted Dr. Sharon Friesen from the Galileo Educational Network to conduct a research study to:

- identify and describe an innovative, accessible classroom;
- describe how digital technologies are and might be used to enable all learners. These technologies include devices, media and services currently on computers or those that could be incorporated to ensure all students are equitably engaged in learning;
- identify and describe the ways in which a teacher uses or might use digital technologies to extend and enrich learning for all students in the regular classroom;
- envision what might be possible in creating an accessible classroom;
- design what an optimal accessible classroom might look like,
- provide recommendations for teachers, schools, school districts and governments on the creation of accessible classrooms; and

[3] See reports http://education.alberta.ca/admin/testing/nationaltesting.aspx

- add to the body of research knowledge and theory about the factors that contribute to the successful accessible classroom

This study found that:

- accessible classrooms are media rich;
- accessible classrooms follow the principles of Universal Design for Learning;
- teachers of accessible classrooms make the curriculum accessible to all learners; and
- accessible classrooms require learning focused networks.

Alberta Education contracted Dr. Friesen to conduct this current study to build on findings from the first study. They were particularly interested in the ways that the four findings would play themselves out in a mathematics classroom.

Context Of This Study

Learning Mathematics in an Accessible Classroom was conducted in a Grade 7 classroom in Foothills School Division over a four-week period beginning at the end of May 2007 and concluding in the middle of June 2007. The research was designed to introduce an intervention into the classroom to determine the extent to which it was possible to make mathematics accessible to all learners. The researchers were particularly interested in students' mathematical proficiency and related achievement when students had multiple ways to:

- access mathematical information and resources;
- express their mathematical understanding; and
- engage with the mathematical concepts.

This study of focused on one aspect of the Grade 7 mathematics curriculum—Geometry. The learning tasks were designed to combine three aspects of the Geometry curriculum strand—measurement, 3D objects and 2D shapes and transformations. Principles of UDL were built into the instructional design. Cognitive supports, mentoring, scaffolding, continuous assessment and collaboration were provided for all students in the design of the social and electronic learning environment of the classroom.

Design of the Study

Purpose and Goals of the Study

The purpose of this research was to investigate and report findings to Alberta Education on:

- the impact of Universal Design for Learning on student mathematics proficiency and achievement ;

- instructional practices that support mathematics learning for all students, particularly those who are identified with special needs.

The goals of the research were:

- to determine the academic achievement of a diverse group of students in a grade 7 mathematics classroom through a statistically valid and reliable pretest and posttest;
- to determine the look and feel for the context of the classroom through videotaping;
- to work with the teacher to design a study based on the principles of Universal Design for Learning;
- to determine the academic achievement of the same group of students through a statistically valid and reliable posttest;
- to provide a visual image of a mathematics classroom that follows the principles of Universal Design for Learning; and
- to add to the research body on how to create effective learning environments for diverse learners.

Research Design and Methodologies

Principles of design-based research informed both the design and the methods used throughout this study. "Design-based research can help create and extend knowledge about developing, enacting, and sustaining innovative learning environments" (The Design-Based Research Collective, 2003, p.5). Design-based research is not a research methodology; rather, it uses both qualitative and quantitative methodologies. It is used when the purpose of the research endeavour is towards sustained innovation, to envision what is not yet, what might be possible in real education settings (Kelly, 2003; van den Akker et al., 2006).

Design-based research is interventionist. Design researchers are trying to make things happen; therefore, there is no claim of objectivity, and the lines between actor and observer are intentionally crossed. "The best design research has a visionary quality that cannot be derived from these other kinds of research, nor does it often arise from practice. It requires a research community driven by potentiality" (Bereiter, 2002, p.324).

The particular strength of design-based research is its ability to increase the capacity of participants to make evidence-based decisions that feedback to change practice while the study is in progress. This is a significant difference from more conventional research designs in which findings emerge primarily at the end of the study when participants have no opportunity to act on them within the context of the stated goals of the project.

Because the capacity to make evidence-based decisions is key to the sustainability of any innovation, design-based research has a unique capacity to reflect the current state of practice and implementation. It also has the capacity to develop participants' insights and abilities to implement chosen goals (Barab and Squire, 2004).

"Design-based research methods focus on designing and exploring the whole range of designed innovations: artifacts as well as less concrete aspects such as activity structures, institutions, scaffolds and curricula" (The Design-Based Research Collective, 2003, p.5 - 6). These methods require close collaboration between the designers of the innovation and the researchers. In this study, researchers and participants were in daily contact. Dr. Friesen did much of the actual classroom teaching, helping by example and by regular debriefing to increase the regular teacher's capacity to design, instruct and assess in ways consistent with the study's vision.

Design-based research is particularly sensitive to local contexts. As researchers, we anticipated that the mathematics study that was the focus of this study would undergo a series of design cycles, enactment, analysis and redesign deeply situated in or growing out of their own context. Thus the initial plans for the geometry study were created with the certainty that how students responded would change subsequent lessons. That is, there was no attempt to implement a study or material created outside the context of this classroom, and these students.

Our challenge was "to develop methodologies which recognize complexities and yet produce robust measures of impact or added value" in order to contribute to the understanding of policy makers (Pittard, 2004, p.181). We did this not by designing a program that could be scaled for delivery across the province. Rather, we have extracted examples and principles for responsive teaching in technology-rich environments that improve mathematical proficiency and achievement for all students, including those identified with special learning needs.

Data Collection

Students whose parents signed consent forms wrote a pretest and post-test. The pretest was administered before the intervention began. The post-test was administered at the conclusion of the four-week intervention. Students were given 45 minutes for both administrations of four sample task geometry related test questions selected from PISA 2000, 2003, 2006: Continent Area, Carpenter, Farms and Twisted Building[4]. While the students in this study were younger than students selected to write PISA mathematics examinations, the researchers were interested in finding test items that were:

- reliability and validity tested (Adams & Wu, 2002);
- designed to assess conceptual understanding and procedural fluency;

[4] These tasks were released in December 2006 in a document called PISA Released Items for Mathematics which can be found at www.oecd.org/dataoecd/14/10/38709418.pdf

- organized contextually in order to facilitate problem solving or strategic competence.

The study was conducted in a rural school board in Alberta. There were a total of 36 students in the classroom. This group of students was composed of nine students, eight boys and one girl, who were (1) coded with social, emotional or academic disabilities and (2) had Individual Program Plans (IPPs) in place. There were also two students who were coded as gifted. These two students also had IPPs in place.

Table 1 shows the eligibility special education code assigned to 11 of the 36 students in this class and the meaning of this code. It also shows the number of students in the class assigned that particular code.

Table 1: Types of Special Codes

Special Education Code	Meaning Of Code	Number of Students
51	Mild Cognitive Disability	1
54	Learning Disability	7
80	Gifted and Talented	2
38	Assigned by School Jurisdiction	1
		11

Table 2 shows that, of the available sample of 36 Grade 7 students, 72% (*N*=26) signed consent forms and produced data from both administrations of the four task questions.

Table 2: Overall Participation

Students	Students in classroom	Signed Consents	Valid pre-test data	Valid post-test data	Valid data for both	Participation rate
Learning Disabilities	9	6	6	6	6	67%
Gifted	2	2	2	2	2	100%
Regular	25	18	18	21	20	72%
	36	26	26	26	26	72%

In addition to the pretest and post-test, observational data was collected from two sources: video observations of classroom interactions and field notes from classroom observations.

Samples of student work were collected throughout the intervention from students whose parents had provided consent.

Researchers interviewed the teacher and the principal. This consisted of a semi-structured interview conducted at the end of the research study. We recorded and transcribed interviews and provided each of the interviewees an opportunity to review and edit their transcript.

Data Analysis

Researchers collected both qualitative and quantitative data. All audio data, observational data, and field notes went through an iterative process of reading, rereading and review. Pre and post-test data were statistically analyzed using SPSS.

Transcripts

Transcripts from interviews were initially read in their entirety to a get a sense of their content and context, without imposing a specific analytic lens. In the second stage, researchers independently read the same text and coded it independently to determine descriptive categories and criteria. We then compared our coding to establish consistency. These were not a priori categories and criteria; rather, they emerged from the analysis of the transcripts themselves. The aim of this level of analysis was to map out the data, review it for further analysis, and become more familiar with its content.

We also analyzed the transcripts to discern patterns of experience. We coded the transcripts, noting all data that related to the patterns. The identified patterns were then expounded on and combined. We defined themes derived from patterns such as conversation topics, recurring vocabulary, recurring activities, meanings, and/or feelings. Themes that emerged from the participants' accounts formed a comprehensive picture of their collective experience. In this way, we were able to establish which themes and sub-themes fit together in a meaningful way (Leininger, 1985).

Observational Analysis

Researchers collected observational data in two ways. Video footage was collected during math classes. The video data was transcribed. One researcher conducted focused observational notes during the 16 classes. These notes were analyzed to discern patterns.

Pretest – Post-test Design and Analysis

The experimental design of this part of the study is a one-group pre-test post-test design. Twenty-seven students were pre-tested. They then participated in the intervention, and then were post-tested at the conclusion of the intervention. Students who did not participate in both the pretest and the post-test were not included in the analysis.

The success of the treatment is determined by comparing the results of the pre-test and post-test scores. The paired-sample t-test for non-independent samples was used to determine if there was a significant difference between the means of each sub-unit of instruction and the total scores. By requiring a higher value to reject the null hypothesis, the t-test makes adjustments for smaller sample size (Gay, Mills, and Airasian, 2006). An alpha level of .05 was used for all statistical tests.

Research Limitations

Limitations of the one-group pre-test post-test design

When the participants do significantly better on the post-test than on the pre-test, with no control-group one cannot assume that the improvement is only attributed to the treatment. For instance, history and maturation of the students is not controlled. The participants may learn how to perform better with each test. Also, participants may have learned something from the pre-test (Gay, Mills, and Airasian, 2006).

In addressing these limitations, qualitative data were used to provide contexts from which to analyze changes in achievement.

The Math Classroom

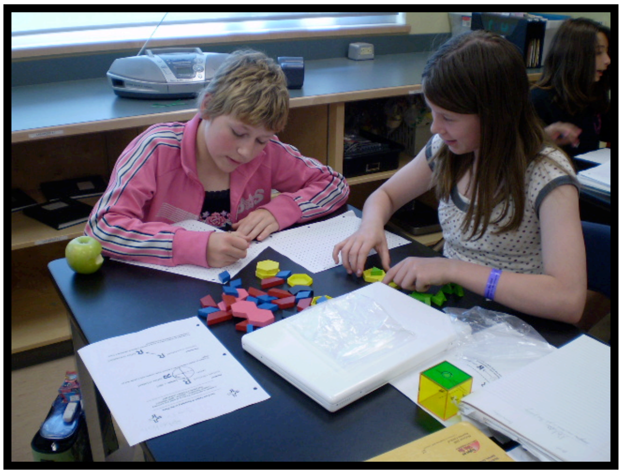

Figure 1: Students

Students
The students in this study represent a fairly homogeneous population in terms of their background. The school is located in a catchment area that butts up against a large urban metropolis. Some students live on farms, while others live on acreages. So while these students live close to a city, they are country kids.

What initially struck us, and many others who visited the classroom, was the large number of students in the class. It wasn't until the teacher started to describe her students that the other part of this picture emerges: the large number of students with identified special needs on IPPs.

Table 3: Comparison of This Class with Alberta Education

Special Education Code	Meaning Of Code	Number of Students in This Class with Identified Special Needs	Number of Students in Alberta 2006/2007 with Identified Special Needs[5]
51	Mild Cognitive Disability	1	7359
54	Learning Disability	7	20 926
80	Gifted and Talented	2	6408
38	Assigned by School Jurisdiction	1	
Overall Total		11 of 36	34 693 of 597 674[6]
Overall Percentage		30.5%	5.8%

[5] As reported by Alberta Education http://education.alberta.ca/admin/special/stats/bycode.aspx
[6] As reported by Alberta Education http://education.alberta.ca/ei/reports/eis1004_2007.pdf

The number of students in this class with identified disabilities is 24.7% above the provincial average. While the provincial data is not disaggregated enough to compare this Grade 7 math class with other Grade 7 math classes in the province, we felt that this group of students, while homogeneous is many ways, represented a good environment in which to work with the principles of UDL.

As we began working with these students we wanted to determine how the students felt about mathematics. A quick survey by a show of hands revealed that a large number of the students in the classroom disliked the subject. They said things like this:

- "*I get it but I don't love it*."
- Many said they couldn't understand math because it made no sense.
- Others couldn't see any use for it: "*Like when am I ever going to need algebra? I've never seen that anywhere except in my math book*."

A small group of students reported that they liked math. Although the majority of students disliked math, many also reported that they thought they were good in math and they got good marks.

Figure 2: Classroom

The Classroom

Getting the classroom ready to accommodate 36 adolescents was a daily undertaking. Students changed classrooms every 45 minutes for different subjects. The class just before this one had 23 only students. Mysteriously, every day tables and chairs disappeared from the room. Before each class the teacher, Mrs. Jamieson,[7] would search throughout the wing of the school for the missing tables and chairs. She would manage to recruit some passing student or teacher to help her carry the furniture into the classroom. Squeezing in a laptop cart of computers, a data projector and card, video cameras and tripods and three additional bodies made for even tighter quarters.

This school had been selected as a pilot site for the first year of a laptop initiative in the school jurisdiction. While the students used the laptop computers in their other courses, the laptop computers were used infrequently

[7] A pseudonym for the teacher is being used in this report.

in the mathematics classroom. Bringing a laptop cart into the classroom during this study added to the congestion and also created some initial confusion for students. What did computers have to do with math?

The Teaching and Learning Prior to this Study

While Mrs. Jamieson has more than ten years of teaching experience, this was her first year teaching mathematics. She volunteered to participate in this study because she saw it as an opportunity to learn. She is an extremely conscientious teacher who cares deeply that her students not repeat her own school math experience.

> *As a student, [I] struggled with math all the way through school and it wasn't until I got to university that it actually started to click for me. But, having said that, sitting in a classroom that has math delivered in ONE* [emphasis hers] *way: "Here it is, now do it, now hand it in and I'll mark it." I'm not sure that that reaches the majority of students. I think it hits middle of the road, most of the kids get some of it, many of them get a little.*

However, Mrs. Jamieson noted that she relied heavily on the math textbook and worksheets to guide the content of her lessons. In describing the ways in which she taught the content she stated:

> *I typically, given that this is my first year teaching math, talk to them about have they done it before, what do you know about it, here is an example, now you do one and then it's a worksheet or something out of a textbook.*

What Mrs. Jamieson has just described is a lesson pattern that has been well documented by Stigler and Hiebert (1999). First identified in the 1995 International Association for the Evaluation of Education Achievement (IEA) Third International Mathematics and Science Videotape Study (TIMSS), this lesson pattern is known as the American teaching script. The lesson pattern repeats in the following way:

- Review previous material
- Demonstrate how to solve problems for the day
- Practice the demonstrated problem
- Correct seatwork and assign homework

In fact, so ingrained was this teaching script, that when teachers started to work with more robust, complex problems in the years after the 1995 TIMSS study, they transformed these "problems designed for teaching rich mathematical concepts into routine procedural exercises"(U.S. Department of Education, 2004).

In addition to a patterned lesson script, theTIMSS 1995 videotape study revealed that U.S[8]. teachers stressed the development of problem-solving skills, while Japanese teachers stressed students' understanding of underlying concepts. More than 60 percent of U.S. teachers specified problem-solving skills as the goal of their lessons. More than 90 percent of Japanese teachers emphasized conceptual understanding over problem-solving ability.

Between the 1995 and 1999 TIMSS studies, the United States applied extensive resources to developing the types of problems that "support mathematical proficiency" (Shanker Institute, 2005). These are problems where students are asked to engage in mathematical reasoning by

- looking for connections,
- developing relationships,
- searching for patterns, and
- making conjectures.

However well constructed the problems, James Hiebert reported that researchers found that in the U.S.

> there are essentially no problems that end up getting worked on in this making connections way. Teachers often step in and do the interesting mathematical work for the students, probably because they sense students are confused or frustrated or getting antsy. And so they think it's their job to step in. It's not that they're trying to subvert students' learning; at least I'm convinced they're not after talking with lots of teachers about this. So this is an important finding for us in trying to understand what kind of teaching is going on in the U.S. and how it differs from high achieving countries. (Shanker Institute, 2005).

In the 1995 TIMSS videotape study, researchers found that in terms of lesson content in the United States, about 90 percent of student seatwork involved practicing routine procedures. In Japan, 41 percent of working time was spent on routine practice and nearly half the time was spent inventing new solutions and engaging in conceptual thinking. Hiebert's recent findings indicate, however, that although lesson content materials have changed in the United States since 1999, the lesson script has remained unchanged. Problems designed to build mathematical proficiency have been turned into routine, procedural exercises.

During our many conversations with Mrs. Jamieson, she frequently noted that this was her first year teaching math. She attributed her style of teaching to this. However, Mrs. Jamieson's mathematics practice resonates strongly with a well-established pattern across North America. Finding ways to break free of this teaching script so that students gain mathematical proficiency by engaging with robust, connections-types of problems is an issue facing most mathematics

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[8] Three countries: U.S., Germany and Japan participated in the 1995 TIMSS videotape study. Canada did not participated the 1995 or 1999 videotape studies.

teachers in the U.S. and Canada.

UDL Design Principles In The Mathematics Classroom

The pervasiveness of the North American script for teaching mathematics provides essential context as we start to focus on the potential for UDL in the mathematics classroom. If fundamental principles of UDL become tied to a teaching script focused on practicing routine, procedural exercises, then mathematical proficiency for all students will not become a reality. Designing mathematics learning for the UDL classroom and teaching mathematics in a UDL classroom suddenly get far more complicated than first imagined, for now it is not just the principles of UDL that need to be brought to bear, but also what research tells us about gaining mathematical proficiency.

This point is of crucial importance in bringing UDL into the mathematics classroom. If the ways in which mathematics is taught involves practicing routine procedures rather than building mathematical proficiency, then it matters little that teachers represent this information in multiple ways or that students have the opportunity to express these routine procedures in multiple ways. The mathematics teaching remains fundamentally problematic.

To relieve the monotony symptomatic of this routine teaching script, the educational community in North America has attempted to make math more fun.

> You'll find one attempt after another, especially recently, to lure children into math by making it fun. 'Manipulables' replace memorizing times tables; a pattern is discovered, and then another; shapes are folded; bells run; numbers dance. All this has the welcome effect of not giving fear and loathing even a look-in: games become the arena where minds encounter math. The problem usually is, however, that these encounters stay superficial—a decorative rather than an architectural instinct is catered to (Kaplan and Kaplan, 2007, p.131).

This is not to say that there is no place for manipulatives in the mathematics classroom. Rather, it is to insist that the mathematics is not contained in the manipulatives, themselves. "For mathematics itself is the study of connections: how things ideally must and, in fact, do sort together—beyond, around, and within us" (Kaplan and Kaplan, 2007, p.5).

A recent longitudinal design-based research study by Swain and Swan (2007) reported that

> some teachers believed that the approaches were about 'standing back and letting the learners discover things for themselves'. … Some teachers became aware of the shortcomings of transmission methods of

> teaching and recognised that 'telling' was not always an effective way of helping learners to understand concepts. Perhaps in reaction to this, they moved to an extreme position of 'not telling' (p.32 – 33).

A teacher working with the principles of UDL in the classroom has the ability to represent any mathematical concept in multiple ways. It is the conceptual understanding, which involves an understanding of concepts, operations and relations, that is focus of the instruction, not the number and types of manipulatives. It is all too easy to slip into decorative forms of mathematics in the name of fun as alternatives to routine procedures. When this happens, the students are frequently abandoned, left to "discover" the mathematics for themselves. The result: students end up with the same lack of mathematical proficiency as those who formerly received an endless repetition of routine procedures.

Teaching for mathematical proficiency (i.e., conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and a productive disposition) requires that the teachers design a learning environment that provides "a solid foundation of detailed knowledge and clarity about the core concepts around which that knowledge is organized to support effective learning" (Donovan and Bransford, 2005, p.569). The type of practice required to promote mathematical proficiency stands in sharp contrast both to transmission-type pedagogies and to discovery-type pedagogies. Rather, the type of practice that builds mathematical proficiency requires that students be brought into a collaborative "relationship between different facts students are learning, between the procedures they are learning, and the underlying concepts" through robust, rich problems and investigations (Shanker Institute, 2005, p.7).

It is to this type of mathematical learning environment that the principles of UDL need to be tethered.

Designing the Geometry Study

The instructional design of the study was created using the Galileo Educational Network's design process. This design process is part of an online professional learning environment called Intelligence Online[9] (IO), so much of this work could be done online with only a few face-to-face meetings with the teacher.

This instructional design process involves

- identifying the understandings that students are to attain,
- designing tasks, activities and lessons to sponsor the stated understanding,
- mapping understandings, tasks and activities to the mandated Program of Studies to ensure that required outcomes are addressed,

[9] IO can be found at www.iomembership.com

- creating assessment to both guide the student learning and to inform students what will count as evidence of learning,
- identifying the ways in which technology will support the learning, and
- identifying the resources.

The IO online environment permits the teacher to publish relevant parts of the planning to an external URL, making the instructional design public. Students had ongoing, online[10] access to the tasks, activities, assessment and resources throughout the study both at home and at school.

We determined that all students should understand that:

1. Geometry is the study of lines, shapes and spaces (objects);
2. Dimensionality of the line (1D) translates into plane geometry (2D) which gives us insight into our home dimension (3D);
3. There are multiple ways to describe geometric forms;
4. There are multiple ways to measure the various geometric forms;
5. Each of the geometric forms has various properties; and
6. The inherent beauty in geometric forms is its symmetry

The goal of a quality design is to develop and deepen student understanding of a particular topic. Students will vary in the types of support systems they will need to do this. Teachers need to vary the ways in which they represent the concepts. Students need the opportunities to express their understanding of the concepts in multiple ways. Teachers and students need to vary or differentiate the ways in which students engage with the concepts through the learning tasks. But the understandings themselves are not differentiated.

We used the following overview of the topics in Geometry to determine the ways in which the topic held together.

[10] The design of the student tasks and activities can be found at www.iostudent.com/1993

Figure 3. Topics in Geometry. (Milgram, 2005, p. 31)

From this we created the following conceptual map of the Geometry terrain so we could see how the concepts were connected. We also used this conceptual map to help us design the learning tasks and map to the relevant General Learner Expectations from the Program of Studies.

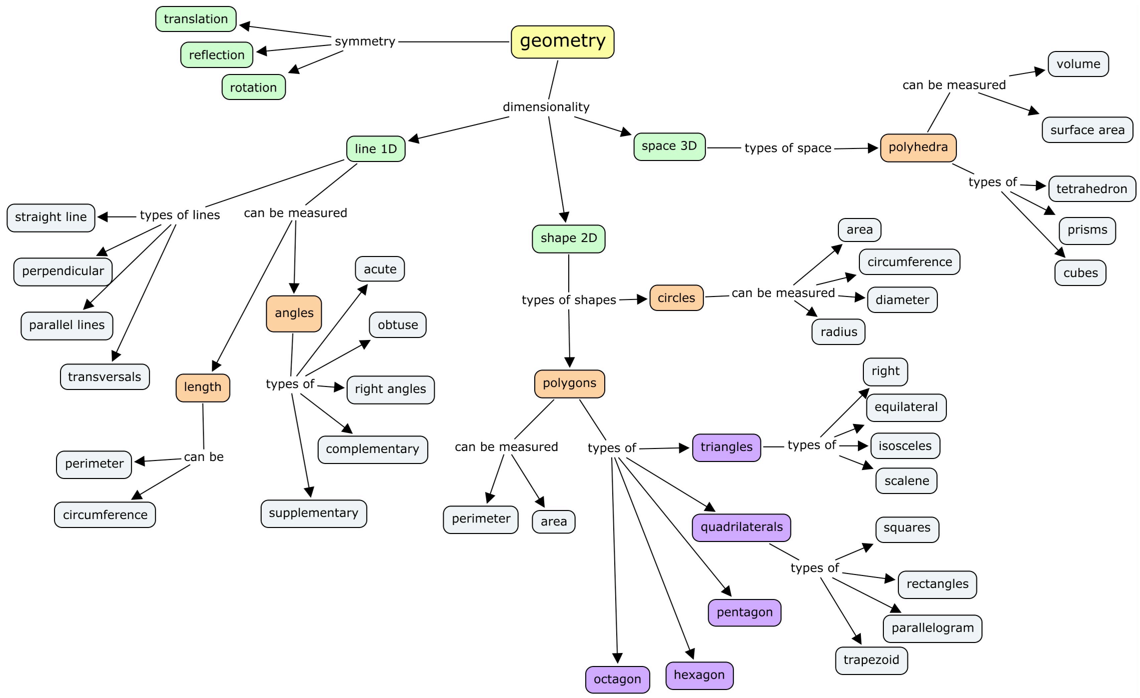


Figure 4. Geometry Conceptual Map

We provided students with a copy of this conceptual map at the beginning of the second week of the study.

Lines, Shapes and Spaces – Task

The first task was *Lines, Shapes and Spaces*. Here is what we presented to the students.

Everywhere we see lines, shapes and spaces. Working with a partner, you will need to:

- Find lines, shapes and spaces in the classroom, school or outside of the school.
- Take digital pictures of a variety of lines, shapes and spaces you find.
- Download these pictures onto your laptop.
- Sort through your pictures and decide on at least two from each category, 1-D, 2-D and 3-D that you want to examine in great detail.
- Now, drag each of the pictures into a Word™ document. (Make sure you save each one with its own name and in a place you can both access.)

Working with your partner, name, describe, analyze and measure the various images you have collected.

Select a way to present your work that best expresses how you understand the ideas in this task.

Lines, Shapes and Spaces - Assessment

The following assessment rubric accompanied this first task.

An Invitation: Lines, Shapes and Spaces				
	Keep Working	**Getting There**	**You've Got It**	**In The Flow!**
Image	Digital photo is blurred or not clear.	Digital photo is clear however the image is vague.	Digital photo is clear with an identifiable image.	Digital photo is clear with a sharply focused image.
Naming of lines, shapes and spaces.	Using own naming structure.	Names lines, shapes and spaces using a mix of standard mathematical conventions and improvised names.	Accurately names lines, shapes and spaces using standard mathematical conventions.	Accurately names lines, shapes and spaces and provides details about which "family" the line, shape or space is part of.
Description of lines, shapes and spaces.	Provides a description using own words.	Provides a description that uses a mix of mathematical terminology and own improvised description.	Provides an accurate description using mathematical terminology.	Provides a detailed description that helps to illuminate features of the line, shape and space.
Analysis of lines, shapes and spaces.	Unable to figure out the various lines, shapes and spaces that compose this figure.	Discerns some of the properties that comprise this figure.	Accurately discerns most of the properties that comprise this figure.	Analysis of this figure helps to illuminate the properties showing how this lines, shape or space relates to other figures.
Measures lines, shapes and spaces	Finds a way to measure some of the lines, shapes and spaces.	Finds different ways to measure the lines, shapes and spaces.	Accurately measures the figure in a variety of different ways.	Accurately measures the figure using a variety of ways and measuring tools.
Teamwork	Conflicts between team members interferes with work quality and production.	Cooperative team work in which team members reinforce each other's learning.	Effective team work in which team members build on and extend each other's ideas.	Effective team work in which team members build on, extend and provide feedback each other's ideas.

Figure 5. Assessment – Lines, Shapes and Spaces

The Alberta Education Mathematics Program of Studies (2007) specifically states that

> the program of studies is arranged into four strands. These strands are not intended to be discrete units of instruction. The integration of outcomes across strands makes mathematical experiences meaningful. Students should make the connection between concepts both within and across strands. (Alberta Education Mathematics Kindergarten to Grade Nine Program of Studies)

However, many teachers find that once they start down the list of General Outcomes accompanied by a long list of Specific Outcomes, thoughts of integrating outcomes is the furthest from their minds. Guided by a long list of outcomes, teachers typically resort to finding ways to directly address each of them in turn. Teachers who might want to integrate the various strands of mathematics have few resources or images to guide their planning, particularly in ways that create high quality mathematics teaching for all students.

As we created the tasks, activities and assessments for this study, we

- integrated the math strands of Number, Patterns and Shape and Space in accord with the intention of the Program of Studies; and
- mapped tasks, activities and assessments to specific outcomes from the Program of Studies. In that way, we ensured that required outcomes were addressed without tethering our instructional design to following a list in sequence, as is so often the case when teachers speak of "covering the curriculum".

Our goal was to create rich, robust math tasks (which could also be called problems or investigations). By this we mean tasks that

- are accessible to all, yet invite students to be challenged to the highest levels;
- invite students to make decisions;
- involve students in speculating hypothesizing, conjecturing, explaining, justifying, proving, reflecting and interpreting;
- promote discussion and questioning;
- encourage originality and invention; and
- have an element of surprise and enjoyment (Swain and Swan, 2007).

A good task, problem or investigation is designed to involve students over a period of time, not just a single lesson. Tasks of this sort can be described as the umbrella under which many other classroom activities cluster. Well designed, these tasks remain constant throughout the unit or study. Inviting tasks, problems or investigations provide the context within which the development of mathematical proficiency becomes meaningful, engaging and enjoyable.

We use the term activities to describe work that engages students at the level of the lesson. Activities build the competencies required to complete the task, solve the problem or investigate the question in mathematically proficient ways. Activities permit teachers to assess, on an on-going basis, the extent to which students understand (or misunderstand) mathematical concepts and procedures. Responsive teaching develops as students' response to one activity shows the teacher what the next step needs to be in order for individuals, small groups or the entire class to progress.

As we started to work with the tasks and accompanying activities, we quickly discovered that all the students needed guidance in how to work with the new demands placed on them by the tasks. Their initial responses indicated that they were not accustomed to:

- working as a collaborative team,
- listening to and building on each other's ideas,
- challenging each other's thinking and conjectures, and
- asking each other to justify their conclusions.

It can be very frustrating for teachers to introduce what they have been told are mathematically better problems only to find that their students have no idea what is being asked of them. We suspect that this frustration accounts, in part, for the tendency of teachers to revert quickly to the conventional teaching script to which they—and their students—are accustomed.

In this study, the teacher was confused and disturbed by students' initial struggles to name, describe, analyze and measure the lines, shapes and spaces they had photographed. Almost as soon as students began to discuss straight lines, for example, they bumped up against a genuinely mathematical problem: how can we prove that a line is actually straight? The task had been designed to create precisely this kind of mathematical "bumping", but the students were confused. It just *was* straight. And what was the problem with just saying that a rectangle has four sides, anyway?

Initially, the teacher interpreted their difficulties in justifying their answers as trouble in the teaching. And she was uncomfortable with rising levels of frustration as students were challenged to push their thinking. Because she had so little experience in teaching mathematics, she could not see how to interpret the students' difficulties or to help them out.

Dr. Friesen had a very different reaction. Seeing that students struggled even to name the lines, shapes and spaces that surrounded them, she recognized that even though they had encountered all the required terms in previous years, perhaps labeling them on worksheets or building them with manipulatives, students had not actually mastered the mathematical concepts that lie behind the labels and the relationships between 1, 2 and 3 dimensions. Asked simply to transfer what they had memorized to the actual world in which they live in a variety of ways, students were at a loss.

And so she adjusted the next set of activities to take students back, helping them build both understanding and enthusiasm for the conundrums raised by seemingly simple questions. It is important here to note, however, the spirit in which she did this, not blaming students for what they "should" have known, which often leads to punishment of the sort that requires students to do pages and pages more of the kind of work that had not served them well in the first place. There was a time in human history when concepts of dimensionality did not exist (at least, not in the form that we now know them). What was the original problem for which geometry became the answer? Why do people still devote their whole lives to its study? And why should these students devote even a minute of their precious lives to learning it, too? (Kaplan and Kaplan, 2007, p. 13).

Discussing curriculum and teaching, Kaplan and Kaplan challenge educators:

> We certainly are still very far from understanding the relation between thought, mathematics, and the world; but our ignorance is no excuse for pretending to others that what took effort (and perhaps even well-prepared luck) should now be obvious to all.

Working with rich, robust tasks requires a particular type of mathematics teaching in order to keep the connections problem up and alive in the classroom.

> The mathematical knowledge needed for teaching is specialized. It's different from the mathematical knowledge required for other mathematically intensive work like physics or accounting or engineering. The mathematical demands of the work are specialized. (Shanker Institute, 2005, p.22).

Most of the students, including those with disabilities, were unfamiliar with this type of mathematics teaching. For the first few days, they were noticeably confused, uncertain, guarded and yet mildly intrigued. Introducing a new way to learn mathematics to these students at the end of Grade 7 and after eight years of schooling that had been primarily focused on learning procedures and right-answer-giving dislocated them temporarily. It wasn't until the second week into the study that we heard, "*This was the coolest project ever. It was cool because I have never done geometry this way before."*

As Kaplan and Kaplan note (2007, p.9), few teachers are themselves taught that math has a history—"as if standing on the shoulders of giants meant we had no need to look down." In order to connect the terms students knew with actual human struggles to make sense of their world mathematically, Dr. Friesen invited them to think about what the words geometry meant.

> T. "*Does anyone know what Geometry means*"
> S1. "*I'll probably be wrong but…*"
> T. "*Go for it*"
> S1. "*Like the area and perimeter of shapes and volume*"
> T. "*Yes, what else*."
> S2. "*Different kinds of shapes and rulers*."
> T. "*A ruler would be a type of a tool that you use in geometry*."

Students continued to list various aspects of geometry. They knew that words such as area, perimeter and volume belonged to such a study. They also knew that shapes such as triangles, rectangles and circles belonged. They named a number of tools such as rulers. However, when prompted to consider what a measure such as area might be good for, or how we could be sure that a line was actually straight, the students were silent. We found this somewhat perplexing, as many of these students were farm kids. Planting, tilling and harvesting were part of these students' experience. The term geometry means, in fact, earth measure.

Looking at the list, it was clear that the students knew many of the words associated with geometry. But we were interested in how they had (or had not) made connections between the geometry they studied at school and their personal lives. That is, we were looking for the ways in which they had transferred their school learning experiences elsewhere:

> Measures of transfer play an important role in assessing the quality of people's learning experiences. Different kinds of learning experiences can look equivalent when tests of learning focus solely on remembering (e.g., on the ability to repeat previously taught facts or procedures), but they can look quite different when tests of transfer are used. Some kinds of learning experiences result in effective memory but poor transfer; others produce effective memory plus positive transfer. (Bransford, Donovan and Cocking, 2000, p. 51).

It was essential to probe the ways these students made sense of the terms that they had just contributed to the list on the board, which by this time they had all dutifully copied into their math notebooks.

Finally a student said, "*I think it was invented by the Greeks*."

"*Why? What do you think the Greeks needed geometry for?*" we asked.

"*To build castles?*" a student responded.

"*Yes, they used it for buildings. For sure. Do any of you know why else they might have needed geometry?*"

Silence.

"*Well let me tell you about the Greeks, the flooding of the Nile, the surveyors….*"

It was Dr. Friesen's opening to tell the long-ago stories of this glorious math topic. Students of mathematics need to know that it has a history that it comes from somewhere. They need to know that the mathematics we encounter today was created by the minds of people to solve problems. Sometimes these problems are practical in nature like resurveying the land after a flood, or building a pyramid or a temple or calculating how much seed a farmer might need to plant an area of land.

Students also need to know that sometimes mathematicians work with problems that appear to have no practical value at all. These types of problems were worked on to discern something further, something quite perplexing, and something worthy of deeper inquiry, such as the length of the hypotenuse when the sum of the other two sides of the triangle equals two.

For this group of students, this time around, Dr. Friesen decided to “back up” by providing a rich context within which to consider issues of the mathematical precision of geometry. The stories were lovely in themselves, and went a long way to relaxing students into an imaginative space in which their struggles and questions become intriguing rather than a sign that you are just no good at math.

Of course, it would be possible to introduce historical stories into classrooms that continue to be dominated by procedural scripts, a problem to which Mighton (2007) alludes. In the same way that mathematical thinking is not contained in manipulatives and models, it is not contained in the stories, either. The teacher’s capacity to create, explore and sustain key connections makes it possible to respond to students’ emerging understandings in ways that are both mathematically and pedagogically sound.

Multiple Means of Representation

Instead of working with predominantly print-based resources that must be modified to meet the needs of individual students, students should have access to “multiple, redundant, and varied representations of concepts and information” (Rose, Meyer and Hitchcock, 2005, p. 25). The term representation refers to the ways teachers organize or formulate content for classroom instruction to present key ideas and concepts to students. For teachers, this is different from knowing the subject of mathematics for their own use; rather, creating or finding appropriate representations means knowing about the discipline of mathematics in ways that make it accessible to students.

> Mathematical ideas can be represented in a variety of ways: pictures, concrete materials, tables, graphs, number and letter symbols, spreadsheet displays, and so on. The ways in which mathematical ideas are represented is fundamental to how people understand and use those ideas. (NCTM, 2000, p.4)

It is essential for teachers (1) to have a wide repertoire of mathematical representations and (2) to know how to establish the equivalence of these representations. Some representations are especially powerful; others, although technically correct, do not open ideas effectively for learners.

Each representation or model also requires different care in use in order to make the mathematical issues salient and usable by students (Cohen, 2005). This applies to all forms of representation: oral, written, pictorial, graphic and symbolic. It is incorrect for teachers to assume that having more representations or models of concepts is all that is needed in the UDL classroom. In fact, just having more might have the opposite effect and end up making the concept more confusing for the students. Mighton (2007) levels precisely this criticism against textbooks commonly available for students. Pages are often a jumble of different ideas, exercises, suggestions, cartoons and models offered in the spirit of providing multiple entry points, but ill-conceived in terms of how such differences relate to one another. Rose, Meyer and Hitchock (2005) caution that, for special needs students, representations that engage an ADHD student could be terrifying for an autistic child.

Rather, what matters most is the teacher's ability to determine the appropriateness and the equivalence of each of representation. In what ways does each representation get at the underlying concept? In what ways do the various representations connect with or relate to each other? If one representation does not work for a student, what are others that would serve better?

> Creating a representation is an act of pedagogical reasoning. Teachers must first turn inward to comprehend the key ideas, events, concepts and interpretations of their discipline. But in fashioning representations teachers must also turn outward. They must try, as it were, to think themselves into the minds of students who lack the depth of understanding they, as teachers, possess. (Wineberg and Wilson, 1991, p.332-333).

To see how this worked in our study we will work with one particular example, circles.

Circles

Initially the students thought we were just being difficult when we pushed them in the naming of circles. After all, it's round, so it must be a circle. *"Really? How can you be sure?"* we asked.

> Learning technical names is sometimes disparaged as a rote activity, but such objections miss the point. Technical names are usually not arbitrary; they encode the conceptual framework in which we organize the things we are naming. (Senechal, 1990, p.145).

What exactly is a circle? Mathematics defines it in a precise way. The idea that a circle is a set of points in a plane that are all an equivalent distance from a given fixed point is not trivial. This definition gives rise to a number of important concepts. In mathematics, names are much more than labels to memorize.

Names, definitions and terms facilitate reasoning about mathematical ideas. We already knew that memorizing definitions had little impact on these students' mathematical understanding. What we asked was for them to look at the shapes they had before them and to determine what made them all circles. The ability to classify geometric forms is essential to Descriptive Geometry and essential to creating mathematical proficiency.

As students started to describe their circles, matters of measurement became important. And this gave rise to a number of misconceptions. One of the first concerned the number Pi (3.16149….): the ratio of the circumference to the diameter of the circle. These students had been using Pi and its symbol π since Grade 3. Now in Grade 7, after being introduced to variables in algebra, they found themselves on shaky ground:

S1*. Is the Number Pi always the same?*
T. Yes it is always the ratio of the circumference to the diameter. Which is always 3.14159…
S2. If the diameter changes does the circumference change?

As you can see in the transcript above, we initially provided the student with a short answer to her question. But then came the second question from another student and then another: "*But what about really big or really small circles?" "Is there a difference between the radius and diameter?" "Why are the formulas for the circumference and the area of a circle different?"*

It wasn't long before we realized that the students had some very strong misconceptions about area, circumference, π, radius, and diameter.

This was one of those places where saw a pervasive misunderstanding of some very fundamental ideas about circles. Both special needs and regular students were asking the same kinds of questions. It was clear that they had memorized terms without also acquiring the ability to reason mathematically with them.

It is at this point that the strength of design-based research becomes apparent. Had the research team maintained a hands-off distance from instruction, they would have observed whether, or how, the teacher was able to adjust her teaching to this new situation. If, as was the case with Mrs. Jamieson, she lacked the experience or background knowledge to take the next steps, the research team might have drawn the conclusion that students of mixed ability founder when asked to engage in sophisticated ways with mathematical concepts. Objectivity of this sort would have missed the very point of research interest.

Design-based research is sensitive to local conditions. It mattered that these particular students did not understand concepts related to circles. How they responded to various activities, their conversations, their confusion and their insights determined how the research team constructed the next day's activities. By closely interacting with the students throughout each of the classes, side-by-side through dialogue, we could hear and see what sense they were making of the various concepts. In that way we could be deeply sensitive to the ways in which they did or did not understand. From what they were saying and doing, we designed our next steps. Among other things, we could see what types of representations we needed to create to help build or strengthen their understanding.

Planning the initial intervention, we had no way of knowing that students would founder conceptually around circles. When it happened, we had to scrap tasks and activities we had planned in advance in favor of building understanding right then and there.

It would be tempting to think of this as back-tracking, or re-teaching and perhaps even to blame the students for forgetting what they had been taught in previous years. But following principles outlined in this report, we welcomed the discovery of areas that required the creation or strengthening of conceptual understanding. While no design process can anticipate *when* such misunderstandings will show themselves, our ways of constructing tasks, activities and assessment ensure that if students misunderstand key concepts, we will see it. In this way, problems of understanding do not become occasions to label students, nor to exclude them from conversations about important ideas. They become, instead, opportunities to provide additional support, representations, conversation and exploration.

And given the importance of circles to geometry, we were confident that every student from gifted to learning or cognitively impaired would find just the right level of challenge to their thinking. No student would be left waiting for the others to catch up.

We designed a number of representations to help the students describe, analyze and measure circles. Our goals in creating these various representations were not so very different from those of the ancient Greeks. We wanted the students (1) to investigate the similarities and differences among shapes and objects, (2) to analyze the components of form, and (3) to recognize different representations of shapes in ways that made sense and created connections.

The three principal, and interrelated tools in geometry are:

- Classification, which is attained through rich descriptions,

- Analysis, which is attained through decomposition, and
- Measurement.

These three tools are also closely tied to symmetry, which is used both to analyze and classify 2D and 3D. We needed all students to understand that about geometry. Had there been students with physical or other impairments, we would have also ensured that the representations we provided made these tools and concepts accessible to them. The point is not to provide all possible representations on every occasion. Rather, it is to exercise pedagogical judgment about what will serve best this time around. Often one representation of the mathematics may be better suited than another to solve the problem or to make an explanation clear.

One of the activities we designed involved using the tools important and available to the ancient Greeks. We had the students construct line segments, construct circles using the length of the line segments, and identify diameters and radii using only a compass and straight edge. For some students this was exactly the representation that they needed to make sense of the concepts related to circles. In fact, these students went far beyond the demands of the activity.

While intrigued with the exercise, some of the students didn't make the necessary conceptual connections. In fact, many students, particularly, but not exclusively, the students with identified learning disabilities, found working with a compass and straight edge quite frustrating. These students didn't have the manual dexterity needed to work with a compass. For other students, the compasses posed a significant problem because the school issue tools lacked screws that would hold the pencils securely.

We had anticipated that this might happen, so we had prepared a similar activity using a dynamic geometry application called First, we knew we would likely need to accommodate learning difficulties. Geometers Sketchpad™ is, in this sense, a powerful AT for special needs students who lack either the manual dexterity or the patience to create precise and accurate constructions. And in other contexts, we have seen teachers give some students access to Sketchpad as an enrichment activity to engage them while the rest of the class caught up.

But the use of this application was more than adapting to learning differences (or even to the poor quality of compasses provided for students). It introduced opportunities for students to explore different representations and to make connections between them. And this is a key element of mathematical proficiency:

> A significant indicator of conceptual understanding is being able to represent mathematical situations in different ways and knowing how different representations can be useful for different purposes. To find one's way around the mathematical terrain, it is important to see how the various representations connect with each other, how they are similar, and how they are different. The degree of students' conceptual understanding is related to the richness and extent of the connections they have made. (Kilpatrick, Swafford and Findell, 2001, p.119).

We also provided the students with the option of using MathsNet (http://www.mathsnet.net/campus/construction/circleonly.html) or Virtual Manipulatives (http://nlvm.usu.edu/en/nav/frames_asid_282_g_3_t_3.html) to work through the various activities.

It is important to keep in mind here that the concepts were not, themselves, changing or being in any sense "dumbed down" in this variety of representation. That is, the learning goals remained constant for all students. Rather, by representing these concepts in different ways, students were given opportunities to use different solution methods, and thus to develop multiple, flexible expressions. The variations in their approaches, solutions and ideas would provide an opportunity for the class to discuss the similarities and differences of the representations and expressions, the advantages of each, and how they must be connected it they yield the same answers.

We had anticipated that (1) some students would come to the conceptual understandings by creating the constructions using compass and straight edge and (2) for others the physical act of creating these constructions would get in the way of developing mathematical proficiency. It quickly became apparent that had we insisted that everyone use only their geometry sets to create the constructions, problems with the tools could have been misread as inability to build conceptual understanding—particularly for those students with identified learning difficulties

All students had the opportunity to work with Geometers Sketchpad™ and many of them chose to do so. Dynamic geometry software provided them with opportunities to:

- create and test out their ideas and conjectures;
- identify what variables were in play;
- speculate about what would happen if they changed those variables;
- create visual proofs, and
- engage in rich, mathematical dialogue with each other to justify and defend their emerging understandings.

The ease with which the software permitted these kinds of investigations provided immediate feedback from the environment, itself—a key factor in learning. Suddenly, those students who struggled with manual constructions were able, as well as the more dexterous, to engage with the ideas of Euclid's postulates. They were no longer relegated to the position of just watching their more conventionally able classmates make progress.

We have included two of the activities we provided for the students, to provide a sense of the various representations the students had available to them. Students downloaded the Sketchpad™ demonstrations and exercises from a website[11] we had created for them.

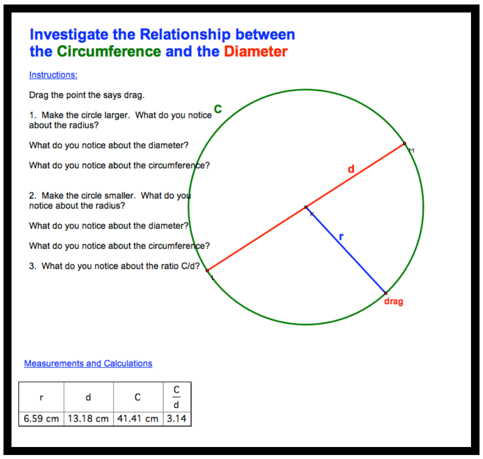


Figure 6. Circumference and Diameter. Demonstration and exercise for students investigating the relationship between circumference and diameter using Geometers Sketchpad™.

[11] This website was in addition to the wikispace and http://www.iostudent.com/1993. However, students could access this website directly from a link in the IO website which they were used to accessing daily.

We placed a number of probing questions for the students directly onto the Sketchpad™ page.

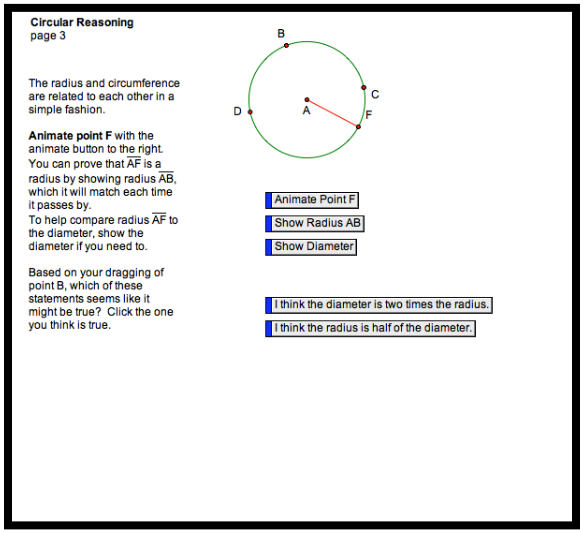


Figure 7. Circle Concepts. Demonstration and exercise for students investigating concepts related to circles using Geometers Sketchpad™. As concepts became more complex, we provided the students with hints and more than one way of exploring the concept through the application. They could choose their own learning path through the investigation.

All students had access to a variety of resources. We identified: Math Open Reference (http://www.mathopenref.com) and WebMath (http://www.webmath.com) as good online resources. We also reminded the students that their textbook was also a good resource; however, none of the students really believed us or took the opportunity to access their textbooks in this way. In fact, one student looked at us, somewhat confused, "*Why do you keep saying that? A textbook is for doing questions, not a resource.*"

We taught alongside the students as they worked through their various problems. In this way we were able to:

- address misconceptions as they arose;
- provide specific, dynamic feedback to guide the students learning;
- provide a different representation should they be having unproductive difficulty with the one they were using;
- make connections between the representation they were currently using and one they had used before or one that the team next to them was using;
- engage students in dialogue to advance their ability to reason mathematically;

- discern what needed to be brought forward to the entire class for discussion;
- determine our next day's activities.

An unexpected outcome of building in such flexibility was that students were initially uncomfortable working with the different representations. As they looked across the classroom, they saw some students using geoboards; others using the virtual manipulatives online site; others using compass, protractor and straight edge; and still others using Geometers Sketchpad™.

They started to ask why their work looked different from others' work, and they were unhappy. If they were doing it right, shouldn't everybody's solutions be the same?

Multiple Means of Expression

Rose, Meyer and Hitchcock (2005, p.35) describe a UDL curriculum as one which provides "flexible models of skilled performance to learn from, opportunities to practice skills and strategies in a supportive environment, relevant and ongoing feedback, and flexible opportunities for demonstrating skill using a variety of media and styles."

From this perspective, schools should provide all students with as wide a range as possible of means to express what they know. In the mathematics classroom, students have not typically had the opportunity to express their thoughts and ideas in multiple ways. As revealed by TIMSS 1995, 1999, students are typically presented with one way to solve problems or practice procedures. Their job is to practice that one procedure until it becomes routine. And it is exactly at this place that we started to hear, "*Can't we just answer questions from our textbook? The work in the textbooks is way easier*" "*I like just answering questions on a worksheet.*" "*How come we have to think?*" "*I don't know what numbers to put into the calculator to get the answer?"*

For many teachers, such growing discontent is disconcerting. In fact, at this point the students themselves can convince a teacher to go back to more familiar, routine procedures. However, procedural fluency within mathematical proficiency does not mean the rehearsal of skills to perform routine procedures. Rather, it refers to (1) knowledge of procedures, (2) knowledge of when and how to use them appropriately, and (3) skill in performing them flexibly, accurately, and efficiently (Kilpatrick, Swafford and Findell, 2001). Such procedural fluency is extremely important.

> When skills are learned without understanding, they are learned as isolated bits of knowledge. Learning new topics then becomes harder since there is no network of previously learned concepts and skills to link a new topic to. This practice leads to a

> compartmentalization of procedures that can become quite extreme, so that students believe that even slightly different problems require different procedures. That belief can arise among children in the early grades when, for example, they learn one procedure for subtraction problems without regrouping and another for subtraction problems with regrouping. Another consequence when children learn without understanding is that they separate what happens in school from what happens outside. (Kilpatrick, Swafford and Findell, 2001, p.123)

Too frequently, in school mathematics, conceptual understanding and procedural fluency compete with each other for attention creating a false dichotomy.

> For example, it is difficult for students to understand multidigit calculations if they have not attained some reasonable level of skill in single-digit calculations. On the other hand, once students have learned procedures without understanding, it can be difficult to get them to engage in activities to help them understand the reasons underlying the procedure. (Kilpatrick, Swafford and Findell, 2001, p.122)

Ball and Bass (2003) demonstrate the ways in which a student solves (expresses), a very ordinary calculation provides the teacher with a great deal of knowledge about how the student understands the concepts related to two digit by two digit multiplication.

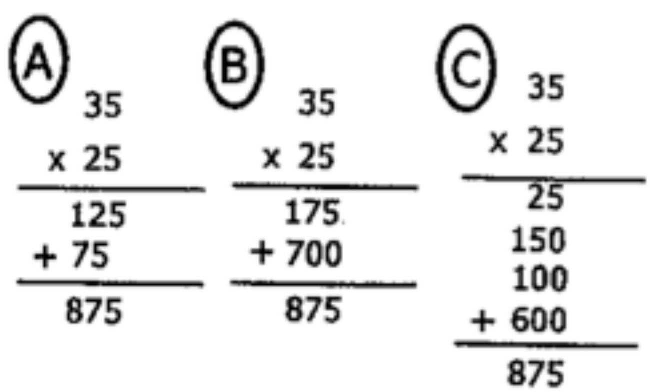


Figure 8: Three different ways to multiply. (Ball and Bass, 2003, p.7).

In these terms, it makes no sense at all to deny any student access to the widest possible expressions of knowledge. And it makes even less sense to restrict what students can do to assignments, work sheets, multiple choice tests and uniform assessment measures that involve the rehearsal and regurgitation of fragmented, memorized bits. Welcoming multiple expressions yields insight into individuals' number sense, not just their procedural fluency. And it provides the opportunity for discussion and debate about the underlying concepts represented by each of the different methods.

That is how understanding is built.

Increasingly, students need to develop sophisticated and multiple ways to express solutions to problems. When one solution is a barrier to knowledge building, whether through an individual's disability or through the inadequacy of the medium to the complexity and richness of the idea under construction, then students need to know that they can find more effective ways to build mathematical proficiency.

This principle lies at the heart of UDL in the mathematics classroom.

But it was not initially easy for the students to live this tension.

During the second week, the students started to realize that we actually wanted to hear how they were making sense of the mathematical ideas. Questions such as "*Tell us how you solved that problem?*" was not a criticism. Instead, it was a question that would be taken up with all seriousness for the benefit of all.

One student solved the problem of finding the area of triangles between two parallel lines with fixed bases this way:

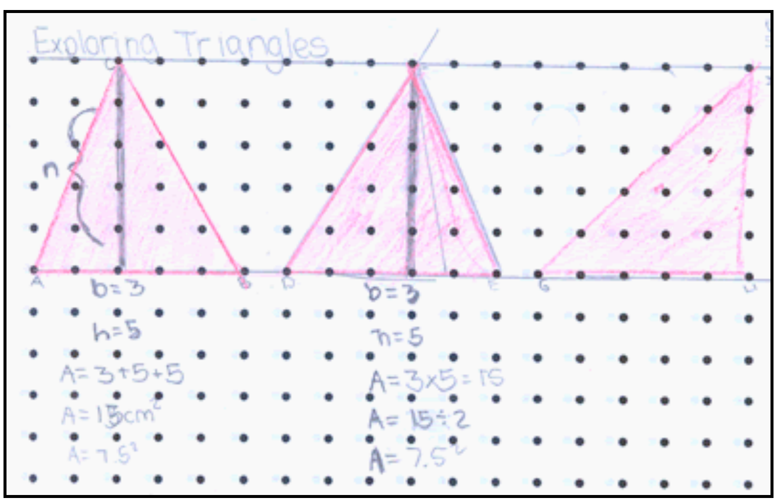


Figure 9: Finding the area of triangles using geoboards.

And another student solved it like this:

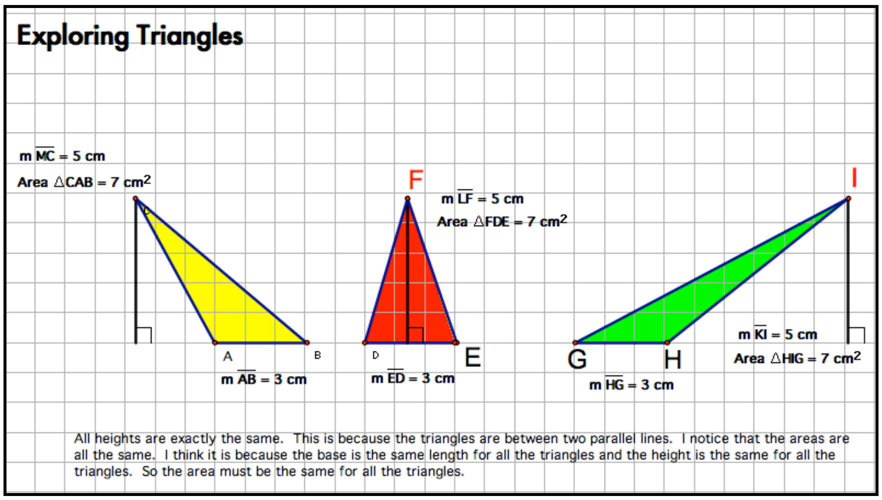


Figure 10: Finding the area of triangles using dynamic geometry application.

Our conversation was about how these two different expressions of the area of triangles were, in fact, similar and connected. We looked at how and why the base in each of these was three and the heights of each of the, now six triangles was five. Once the students were are able to see this, they were able to focus on why and how the area for all the triangles had to be the same.

> *They recognize that the process is as important as the right answer. They now understand that what I can prove that is, as right as what you are saying or I understand why that is right. I think that is more valuable and I think that's the 'stuff' that's going to make a difference for them in geometry down the road.* (Teacher interview)

In working with technology-rich, multiple and varied representations and expressions, we wanted students to recognize that a mathematical community is one in which differences are valued because of the opportunities they provide for explanations, justification, debate and exploration. Being good at math had come to include care and respect for others: listening, hearing, seeing; collaborating, building on, and challenging each others' ideas.

Being good at math was starting to mean far more, now, than finding the right answer quickly and in the way that everyone else found it, too.

Upsetting The Natural Order

During these class discussions, particularly towards the end of the second week, something quite unexpected surfaced. As we anticipated, it was becoming increasingly difficult to identify the nine students who were "coded" as having learning disabilities. They accessed the same learning curriculum, were given the choice and support to express their learning using methods of their choice and they were not isolated from the their peers.

Many of the questions during the open discussion times came from the "coded" students. Concepts such as Pi, circumference, area and perimeter intrigued them. They questioned with confidence, added their thoughts to the general discussions, and became active participants in their own learning. That is, students who typically had difficulty understanding mathematics, those for whom mathematics typically didn't make much sense, started to speak up.

What we did not anticipate was the extent to which their increasing proficiency temporarily bothered some of the students who saw themselves, or were seen by others as being good at math. They protested: "*Hey, how do you know that?*" "*You don't get math*." Then in quiet, hushed conversations, out of what they thought were the ears of the teacher and researchers, their complaints grew.

It wasn't fair that the coded kids were getting math. Fortunately, this attitude changed as they realized that learning in this way was not a competition. Too often, when marks are used as a sorting device, achievement becomes a zero sum game: the advance of some is gained at the expense of others. As students' engagement increased, they lost this fear.

Initially, the academic order of the classroom was disrupted, a social order created by conventional educational structures and processes and their organizing principles and assumptions. It is beyond the scope of this study to determine whether the reactions of these students would be in any way typical of other classrooms when those who "everyone knows" are left behind suddenly emerge as equally able and engaged. For now, however, we are comfortable noting two things.

First, "equality is—and must continue to be—a key goal of any public education system"; however, "we need new ways of thinking about equality, ways that do not involve sameness, or one-size-fits-all approaches" (Gilbert, 2005, p.102). And it would appear that one of those new ways of thinking may involve re-interpreting the unintended consequences of meeting individual needs by identifying some students as inherently less able than others when, in fact, their perceived disabilities are to some extent artifacts of our own structures and pedagogies. UDL has the potential and the possibility of being one of these new ways.

Second, it may be important to examine the history of mathematics as a gatekeeper subject that has traditionally been used to separate the academically able sheep from the less talented goats. Some members of the community of mathematics educators and researchers (Mighton, 2007; Kaplan and Kaplan, 2007; Swain and Swan, 2007) have been working to expose what they call the myth of mathematical talent as the primary way to explain why some students “get” math and others do not. At a recent conference at the Mathematisches Forschungsinstitut Oberwolfach, Germany, Dr. Friesen observed the emergence of educational policies in several countries aimed to identify and stream the mathematically capable from those deemed as early as Grade 3 to be incapable of understanding pure or academically-oriented mathematics.

We began this report with the account of a conversation with an individual from Alberta Education concerned that Alberta not stand still in terms of improving mathematics education in the province. Here, we would note how striking is the Canadian (and Albertan) concern to give as many students as possible access to robust learning experiences that develop genuine mathematical proficiency. Our study demonstrates that in a short time, even in a classroom with a disproportionately large number of coded students, it is possible to raise both ceiling and floor simultaneously.

Choosing to do so becomes, then, a matter of policy. How badly do we want equality for all students, and are we prepared to weather what may be inevitable storms from the highest levels right down to the playground?

Multiple Means of Engagement

Students who are engaged have volunteered to give their attention to and interest to the learning task (Rose and Meyer, 2002).

In this study, we were interested in looking at the extent to which students became engaged and volunteered to give their attention and interest to geometry. The examples contained in this report demonstrate what such engagement looks like.

Engagement is not an attribute of the individual (as in, Alice has problems engaging with her school work), although many teachers think of it in this way. Nor is the ubiquitous “time on task” that so often results in teaching-as-management. In other contexts, we have had experience of teachers who design or assign work for students, then monitor whether or not they are doing what they are supposed to. We worry, in fact, that an unintended consequence of emphasizing multiple forms of representation could be that such teachers increase the numbers and types of activities as their main way of “doing” UDL.

Engagement doesn't mean answering the even numbered questions or trying every suggestion in the textbook. Rather, it seems to have at least two attributes we saw emerge in this class:

- Increased willingness to spend time exploring a question or idea.
- Increased enthusiasm for ambiguity and uncertainty.

Fast-right-answer-giving works against both elements of engagement. When students come to see mathematics as arriving quickly at the same answer in the same way as everyone else, differences in approach or challenges from others are seen as error.

As Kaplan and Kaplan note, (2007, p. 48),

> mathematics is pervaded by ambiguity. For what is an equation but the confronting of two points of view? To say that the square on the hypotenuse is also the sum of the squares on a right triangle's two legs is the beginning, not the end, of deep insight.

Thus, in our earlier examples of the definition of a circle, or finding different expressions for the area of a triangle, the precise definition of a term is not simply the application of a label, nor is the calculation of area only the application of a memorized procedure. Discovering how concepts are related, how things apparently different are connected by "startling unities behind the flicker of appearances" (Kaplan and Kaplan, 2007, p. 48) catches our attention and engages our interest. "How can this be?" a student might ask. Or, "I never knew circles and triangles were connected!" This, we would argue, is characteristic of mathematical confidence and proficiency.

Too often students with learning difficulties collapse under the strain of the unexpected. For them, ambiguity is a threat, not a charm. What is needed is what Kaplan and Kaplan (p. 48) call "attention without tension":

> walking through an inviting landscape, taking in its foggy valleys and cloudy peaks, pausing for views that seem to unify and views where everything falls or rises away. Like good explorers, we're willing to put up with a bit of uncertainty in our situation for the adventure of it all….

Student interest and engagement increased in precisely these ways throughout the study. Digital media engaged the students and although somewhat distracting in the beginning it was the challenge, support and flexibility of the tasks that eventually engaged the students.

A Matter of Time

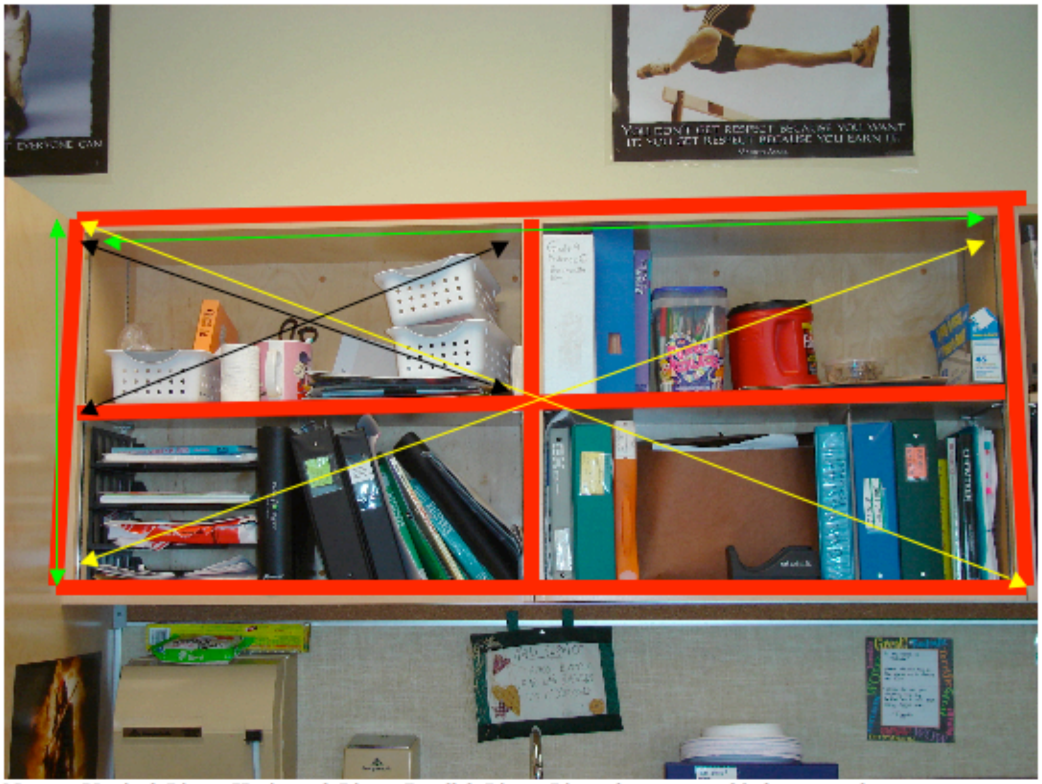
The constraints of 45 minute blocks of time became evident very early in the study. No sooner had students settled into productive work than the bell rang, signaling a change of class. Frustration started to build for both the students and members of the research team.

The tasks and activities required time. The students needed time to dialogue with each other, to explore concepts in depth, to think and reason without interruption and to test conjectures and justify solutions. By the time students settled in from their previous class, they were left with approximately 30 minutes to become engaged in a demanding math task.

> The effectiveness of mathematical teaching and learning is a function of teachers' knowledge and use of mathematical content, of teachers' attention to and work with students, and of students' engagement in and use of mathematical tasks. Effectiveness depends on enactment, on the mutual and interdependent interaction of the three elements—mathematical content, teacher, students—as instruction unfolds. The quality of instruction depends, for example, on whether teachers select cognitively demanding tasks, plan the lesson by elaborating the mathematics that the students are to learn through those tasks, and allocate sufficient time for the students to engage in and spend time on the tasks. (Kilpatrick, Swafford and Findell, 2001, p.8-9).

At the beginning of the second week, Mrs. Jamieson changed the schedule so they now had 90 minute blocks of time for mathematics. The research team initially thought that the students might protest as this meant giving up a 15 minute personal break. However, there were no protests. With the additional gift of time, the research team observed students' increased willingness to spend time exploring a question or idea and increased enthusiasm for ambiguity and uncertainty. They repeatedly saw the depth of thinking and understanding evident in the following (Figure 11) artifact of students' work.

Lines



Names: Vertical Lines, Horizontal Lines, Parallel Lines, Lines that meet at 90 degree angles

Description: The vertical line runs along both outsides of the shelf and inside, along the divide that separates the shelves. We wanted to find out whether the lines were straight, so we measured the distance between each of the sets of vertical lines and found them to all be the same distance apart. Therefore the vertical lines are all straight lines. It doesn't look that way in our picture but that is because of the angle of our camera to the shelves.

The horizontal line runs along the top, the bottom and right through the middle of the shelves to once again divide them from the others. Again, both lines are an equal distance apart throughout their span which means that they are parallel and therefore straight.

Analysis: We discovered that the vertical and horizontal lines meet at 90 degree angles throughout the entire construction. In fact, the lines meet forming rectangles in the 2D plane and a rectangular prism in 3D. Diagonal lines can be drawn from vertex to vertex in each of the rectangles showing that these can be decomposed to form 4 triangles. Now, we also found that we could do that with the whole shape as well. So there are not only vertical and horizontal lines, there are also diagonal lines in this combination of multiple lines. It runs both right and left through the center, connecting from the top corners and then meeting again at the bottoms.

Figure 11: Student work - Lines

Gaining mathematics proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition) using multiple, flexible means of representations and multiple, flexible means of expression requires sustained blocks of time longer than 45 minutes.

Assessment

The value of assessment *for* learning, that is formative assessment, is becoming more widely recognized (Black and Wiliam, 1998; Davies, 2002-2003; Stiggins, et al, 2004; Bransford, Donovan and Cocking, 2000).

> Professors Paul Black and Dylan Wiliam synthesised evidence from over 250 studies linking assessment and learning. The outcome was a clear and incontrovertible message: that initiatives designed to enhance effectiveness of the way assessment is used in the classroom to promote learning can raise pupil achievement (Assessment Reform Group, 1999, p.4).

In their 1998 study, Black and Wiliams found evidence to show that assessment practices which increased pupil achievement were weakly developed in the majority of classes. They attributed this to three factors:

- the assessment methods that teachers use are not effective in promoting good learning
- marking and grading practices tend to emphasize competition rather than personal improvement
- assessment feedback often has a negative impact, particularly on pupils with low attainments who are led to believe that they lack 'ability' and are not able to learn (Black, 2004, p.1)

Black (2004) reports that much confusion exists regarding the type of assessment that actually increases student learning and achievement. He notes that misunderstandings arise when teachers think that teacher-made tests and student portfolio assessments can be used to improve student learning. They cannot because they are put together at the end of a piece of learning. Assessment for learning occurs while the learning is taking place.

In designing the pedagogical interactions in this study, researchers placed a strong emphasis on assessment for learning, defined as

> any assessment for which the first priority in its design and practice is to serve the purpose of promoting pupils' learning. It thus differs from assessment designed primarily to serve the purposes of accountability, or of ranking, or of certifying competence. An assessment activity can help learning if it provides information to be used as feedback, by teachers, and by their pupils in assessing themselves and each other, to modify the teaching and learning activities in which they are engaged. Such assessment becomes 'formative assessment' when the evidence is actually used to adapt the teaching work to meet learning needs. (Black, et al. 2002, p.2-3).

The research team employed various forms of assessment for learning:

- Sustained dialogue with students. A great deal of effort went into examining the geometric mathematical territory by Dr. Friesen so she could engage students in dialogue around questions that were worthy mathematically and which could assist students in making connections, developing reasoning and building mathematical proficiency.
- An analytic trait rubric made available to the students before the study started and constantly available to the students throughout the study at (www.iomembership.com/1993).
- Specific daily feedback, both written and oral from the research team and from each other around identified criteria that were open and readily available.
- The dynamic geometry application also provided the students feedback. Students got immediate feedback from the dynamic geometry application when a construction wasn't working as they had intended it to.
- Daily reflections by the students, Mrs. Jamieson and the research team members.

The research team created a number of environments and ways to record the students' written work. Because so much of that work was digital, the research team created a wiki (http://gr7math.wikispaces.com). They also requested that the jurisdiction's information technology department provide the students with email accounts. In this way, members of the research team could provide the students with additional written feedback and respond to the students' reflections.

However, access to school email both at school and at home was less than desirable. While most of the students were able to successfully set up and access their email account, the school network was so slow that sending attachments was impossible. Without the ability to attach documents to email, the research team abandoned the idea of using email. Not daunted, the research team decided that they would have the students save their documents into a common shared folder from which the research team would then download the documents after class. However, this too failed, as it took too long for the students to save the documents into the common folder on the network. Next, they attempted to have students print their documents. No go most days! The printers in the school did not have enough toner to create a clear enough image to read. So the research team focused strongly on providing strong, ongoing oral and written feedback individually and to small groups. Dr. Friesen also built a large group conversation into each day so students could test out their ideas, listen to other students emerging understandings, build on other students' ideas and seek clarification.

The analytic trait rubric that was used to guide the student learning was also used at the end of learning as summative assessment. Students worked with their team members to assess their final task performance. Having worked with the rubric throughout the learning task, they knew what criteria they were working towards. Students were able to accurately self assess their work. Both student self-assessment and researcher/teacher assessment placed most students in the top two levels.

An Invitation: Lines, Shapes and Spaces

	Keep Working 1	Getting There 2	You've Got It 3	In The Flow! 4
Image	Digital photo is blurred or not clear.	Digital photo is clear however the image is vague.	Digital photo is clear with an identifiable image.	Digital photo is clear with a sharply focused image.
Naming of lines, shapes and spaces.	Using own naming structure.	Names lines, shapes and spaces using a mix of standard mathematical conventions and improvised names.	Accurately names lines, shapes and spaces using standard mathematical conventions.	Accurately names lines, shapes and spaces and provides details about which "family" the line, shape or space is part of.
Description of lines, shapes and spaces.	Provides a description using own words.	Provides a description that uses a mix of mathematical terminology and own improvised description.	Provides an accurate description using mathematical terminology.	Provides a detailed description that helps to illuminate features of the line, shape and space.
Analysis of lines, shapes and spaces.	Unable to figure out the various lines, shapes and spaces that compose this figure.	Discerns some of the properties that comprise this figure.	Accurately discerns most of the properties that comprise this figure.	Analysis of this figure helps to illuminate the properties showing how this lines, shape or space relates to other figures.
Measures lines, shapes and spaces	Finds a way to measure some of the lines, shapes and spaces.	Finds different ways to measure the lines, shapes and spaces.	Accurately measures the figure in a variety of different ways.	Accurately measures the figure using a variety of ways and measuring tools.
Teamwork	Conflicts between team members interferes with work quality and production.	Cooperative team work in which team members reinforce each other's learning.	Effective team work in which team members build on and extend each other's ideas.	Effective team work in which team members build on, extend and provide feedback each other's ideas.

Comments:

Figure 12: Student Summative Self Assessment

Achievement As Measured By A Standardized Test

One of the goals of this research study was to determine the academic achievement of a diverse group of students in a Grade 7 mathematics classroom through a statistically valid and reliable pretest and to determine the academic achievement of the same group of students through a statistically valid and reliable post-test. As indicated previously in this report, the research team selected four task questions from PISA 2000, 2003, 2006: Continent Area, Carpenter, Farms and Twisted Building[12]. While the students in this study were younger than students selected to write PISA mathematics examinations, the researchers were interested in finding test items that were

- reliability and validity tested (Adams & Wu, 2002);
- designed to assess conceptual understanding and procedural fluency;
- organized contextually in order to facilitate problem solving or strategic competence.

PISA tests are designed to test various mathematical competencies:

- Mathematical thinking and reasoning
- Mathematical argumentation
- Mathematical communication
- Modeling
- Problem posing and solving
- Representation
- Using symbolic, formal and technical language and operations
- Use of aids and tools

"PISA does not use tasks that access the above competencies individually. When doing 'real mathematics' it is necessary to draw simultaneously upon many of these skills" (OECD, 2000, p.83).

The following is an analysis of student performance on each of the task questions and an analysis of their total performance on the pretest and posttest. Our analysis consisted calculating the mean, standard deviation and standard error. We compared the pretest to the posttest on each of these measures. To determine the size of the variation within the group of students taking the same test we ran a paired-sample t-test for non-independent samples. This was used to determine if there was a significant difference between the means of each sub-unit of instruction and the total scores. The *t*-test tests the statistical significance of the difference in the two means. Specifically, instead of treating each group separately, and analyzing raw scores, the paired-sample t-test for non-independent samples allowed us to look only at the differences between the two measures, the pretest and posttest for each of the groups of students to

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[12] These tasks were released in December 2006 in a document called PISA Released Items for Mathematics which can be found at www.oecd.org/dataoecd/14/10/38709418.pdf

determine whether we had a statistically significant different in achievement between the pretest and posttest results. By subtracting the first score from the second for each subject and then analyzing only those "pure (paired) differences," we were able to exclude the entire part of the variation in our data set that results from unequal base levels of individual students. By requiring a higher value to reject the null hypothesis, the t-test makes adjustments for smaller sample size (Gay, Mills, & Airasian, 2006). The researchers selected an alpha level of .05 for all statistical tests.

1. *Task One: Continent Area task*
   The Continent Area task requires students to identify an appropriate strategy and method for estimating the area of an irregular and unfamiliar shape, and to select and apply the appropriate mathematical tools in an unfamiliar context. Students need to choose a suitable shape or shapes with which to model the irregular area (for example, approximating parts of the map with rectangle(s), circle(s), triangle(s). Students need to know and apply the appropriate formulae for the shapes they use; to work with scale; to estimate length; and to carry out a computation involving a few steps.

   Table 4 below shows how the students performed on the *Continent Area* Task pretest and posttest.

Table 4: Mean Scores for Continent Area Task

Paired Sample - Continent Area task	*N*	Mean	Std. Deviation	Std. Error Mean
LD Coded Pretest	7	1.00	0.82	0.31
LD Coded Posttest	7	1.29	0.95	0.36
Not-Coded LD Pretest	20	1.10	0.85	0.19
Not-Coded LD Posttest	20	2.10	1.33	0.30
All Students Pretest	27	1.07	0.83	0.16
All Students Posttest	27	1.89	1.28	0.25

The graph below (Figure 13) shows an increase in mean scores between Continent Area task pretest and posttest.

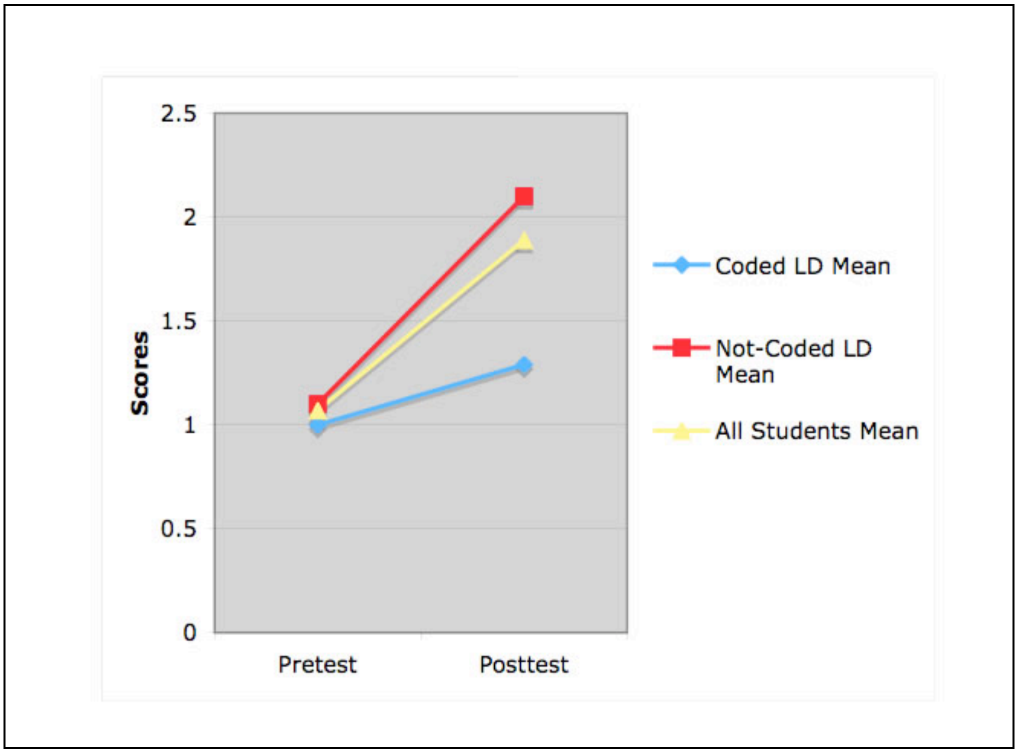

Figure 13: Mean Scores for Pretest-Posttest for Continent Area task

The data in Table 5 below indicates that the mean on Continent Area task scores was significantly different for Not Coded LD student and for all students. However there was not a significant difference between the pretest and posttest scores for the Coded LD students. This difference on the pretest and posttest scores represents a significant improvement in achievement for the Not Coded LD students and for the class as a whole.

Table 5: Paired Sample *t* Test Results for Continent Area Task

Paired Sample *t*-test	*t*	*df*	Sig. (2 tailed)	Mean Difference
Coded LD	-1.00	6	0.36	-0.29
Not Coded LD	-3.34	19	0.00	-1.00
All Students	-3.41	26	0.00	-0.82

The Continent task is a particularly difficult, demanding a high level of competency, including the ability to calculate area by using scales. In the PISA test, this task is given a difficulty level of 712, one of the highest levels assigned to tasks. The researchers are impressed that all students demonstrated at least some degree of increased ability to perform such a demanding task after the four-week study.

During the study, some students worked with scale while measuring some of the images they took. Many students did not. Many students chose to measure the actual object rather than determine the scale represented on their image. In addition to scale, this task required students to calculate the area of an irregular shape—the continent of Australia. Students were required to analyze this shape by decomposing it into constituent shapes: triangles, rectangles and squares to calculate the area. Once this was accomplished they needed to calculate measures of all the constituent shapes adding them altogether to arrive at a reasonable estimate of land area.

As the following *Farm Task* indicates, the Coded LD students developed proficiency in calculating area; however they may have experienced confusion in recognizing that all the smaller shapes needed to be added together to determine the whole. When examining the actual tests, researchers noted that all students divided the irregular landmass into smaller shapes. On further examination of the actual tests, it is obvious that scale caused these students difficulty.

2. Task Two: *Farm Task*
   Students are given a mathematical model (in the form of a diagram) and a written mathematical description of a real-world object (a pyramid-shaped roof) and asked to calculate one of the lengths in the diagram. This task requires students to work with a familiar geometric model and to link information in verbal and symbolic form to a diagram. Students need to visually "disembed" a triangle from a 2-dimensional representation of a 3-dimensional object; to select the appropriate information about side length relationships; and to use knowledge of similar triangles in order to solve the problem.

   Table 6 below shows how the students performed on the Farm Task pretest and posttest.

Table 6: Mean Scores for Farm Task

	N	Mean	Std. Deviation	Std. Error Mean
LD Coded Pretest	7	0.57	0.54	0.20
LD Coded Posttest	7	1.14	0.69	0.26
Not-Coded LD Pretest	20	0.95	0.83	0.19
Not-Coded LD Posttest	20	1.15	0.59	0.13

All Students Pretest	27	0.85	0.77	0.15
All Students Posttest	27	1.15	0.60	0.12

The graph below (Figure 14) shows an increase in mean scores between Farm task pretest and posttest.

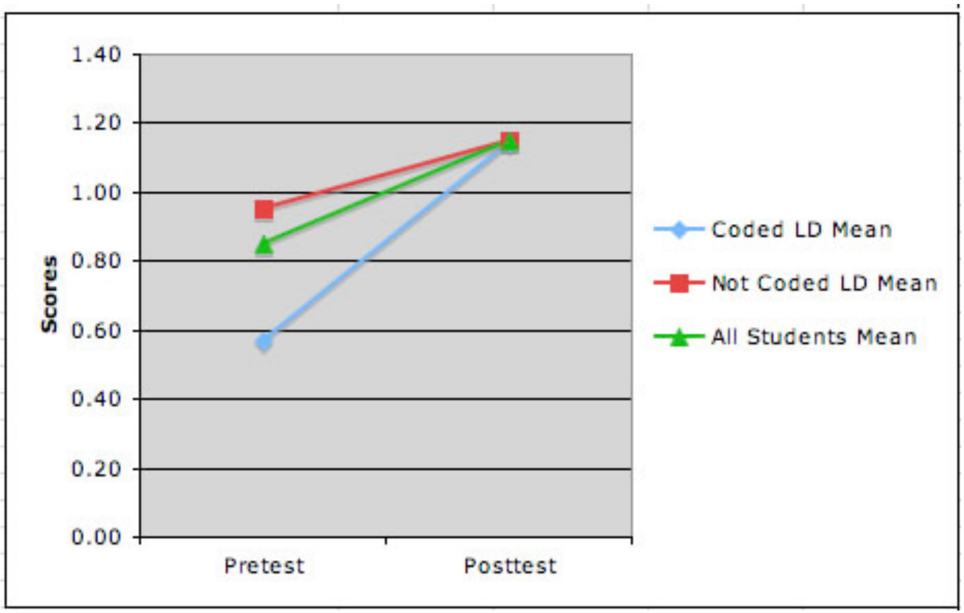

Figure 14: Mean Scores for Pretest Posttest for Farm task

The data in Table 7 below indicates that the mean of scores on the *Farm* task is significantly different for Coded LD students, but not for Not Coded LD students. This represents a significant improvement in achievement for the Coded LD students, but not the Not Coded LD students. Overall, the class did not demonstrate significant improvement.

Table 7: Paired Samples *t*-test for Farm Task

Paired-Sample *t*-test	*t*	*df*	Sig. (2 tailed)	Mean Difference
Coded LD	-2.83	6	0.03	-0.57
Not Coded LD	-1.07	19	0.30	-.20
All students	-1.99	26	0.06	-.30

This task receives a task difficulty level of 492 on the PISA test. 492 is within the middle range of task difficulty. While still demanding for 12 year olds, this was a more familiar task given the type of tasks and activities students had engaged in throughout the study. It is clear that all students were able to transfer the learning gained during the study into this new context. What is worth attending to is the level of performance attained by Coded LD students. These students attained a level of performance comparable to the Not Coded LD students.

3. Task 3: *Carpenter Task*
   This task requires students interpret and link text and diagrams representing a real-world situation; show insight in 2-D geometrical properties; extract information from geometrical representation; calculate perimeters for compound and irregular shapes; apply routine procedures.

   Table 8 below shows how the students performed on the *Carpenter Task* pretest and posttest.

Table 8: Mean Scores for Carpenter Task

	N	Mean	Std. Deviation	Std. Error Mean
LD Coded Pretest	7	1.29	0.76	0.29
LD Coded Posttest	7	3.14	1.22	0.46
Not-Coded LD Pretest	20	2.00	1.08	0.24
Not-Coded LD Posttest	20	2.65	0.93	0.21
All Students Pretest	27	1.81	1.04	0.20
All Students Posttest	27	2.78	1.01	0.20

The graph below (Figure 15) shows an increase in mean scores between Carpenter task pretest and posttest.

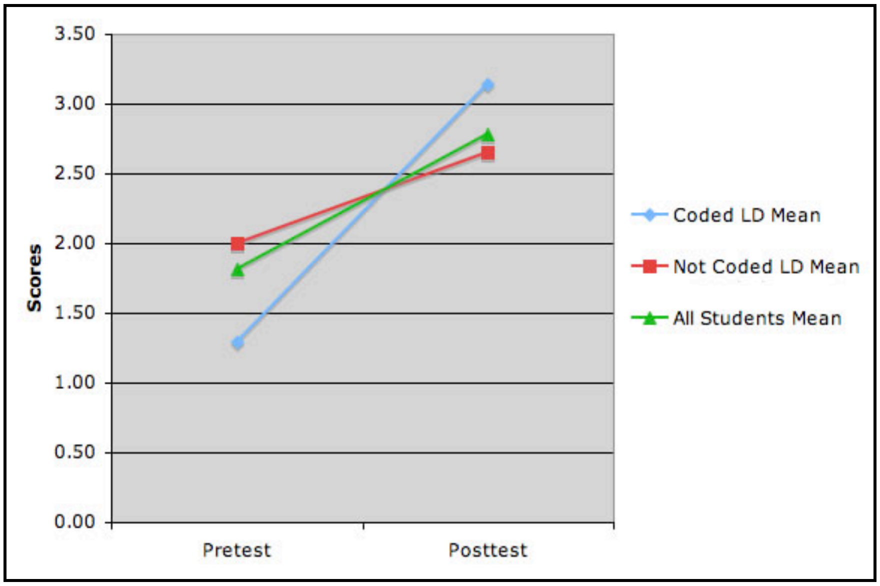

Figure 15: Mean Scores for Pretest Posttest for Carpenter task

The data in Table 9 below indicates that the mean of *Carpenter Task* scores is significantly different for the Coded LD students, for Not Coded LD students and for all participating students. This represents a significant improvement in achievement for the Coded LD students and the Not Coded LD students. Overall, the class demonstrated significant improvement.

Table 9: Paired Samples *t*-test for Carpenter Task

Paired-Sample *t*-test	*t*	*df*	Sig. (2 tailed)	Mean Difference
Coded LD	-4.04	6	0.01	-1.86
Not Coded LD	-2.80	19	0.01	-0.65
All students	-4.20	26	0.00	-0.96

This task receives a task difficulty level of 687, at the highest level within the middle range of task difficulty. Like the *Farm Task*, this is a demanding task for 12 year olds, requiring a high degree of mathematical proficiency. Again it is clear that the tasks and activities students had engaged in throughout the study helped them develop this proficiency. While students did not work on tasks like this one, they did calculate perimeters for compound and irregular shapes and applied routine procedures. The Coded LD students' performance on this question shows that they can reach high levels of mathematical proficiency.

4. Task Four: *Twisted Building Task*
   This task requires students to imagine the cumulative effect of the twisting phenomenon over a number of steps and to construct a graphic representation of those turns. They are required to extract information from geometrical representation; calculate degrees of rotation and determine orientation following a number of turns.

   Table 10 below shows how the students performed on the *Twisted Building Task* pretest and posttest.

Table 10: Mean Scores for Twisted Building Task				
	N	Mean	Std. Deviation	Std. Error Mean
LD Coded Pretest	7	1.86	1.46	0.55
LD Coded Posttest	7	4.00	0.00	0.00
Not-Coded LD Pretest	20	2.20	1.94	0.43
Not-Coded LD Posttest	20	4.00	0.00	0.00
All Students Pretest	27	2.11	1.81	0.35
All Students Postest	27	4.00	0.00	0.00

   The graph below (Figure 16) shows an increase in mean scores between *Twisted Building Task* pretest and posttest.

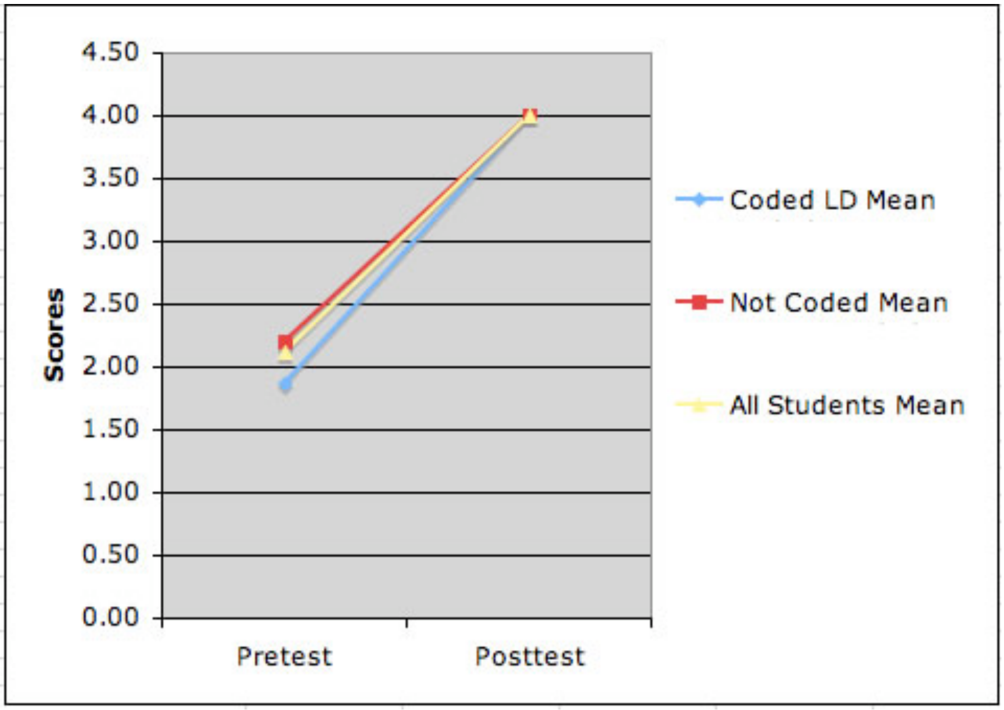

Figure 16: Mean Scores for Pretest Posttest for Twisted Building Task

Table 9 below indicates that the mean of *Twisted Building* task scores is significantly different for Coded LD students, for Not Coded LD students and for all participating students. This represents a significant improvement in achievement for the Coded LD students and the Not Coded LD students. Overall, the class demonstrated significant improvement.

Table 11: Paired Samples t-test for Twisted Building Task

Paired-Sample *t*-test	*t*	*df*	Sig. (2 tailed)	Mean Difference
Coded LD	-3.87	6	0.01	-2.14
Not Coded LD	-4.16	19	0.01	-1.80
All students	-5.44	26	0.00	-1.89

The researchers were unable to find a task difficulty rating for this task in the 2003 PISA Technical Report. However, we estimate it to be at the top end of the middle range of difficulty. It is a demanding task for 12 year olds, requiring a high degree of mathematical proficiency. Again, tasks and activities students had engaged in throughout the study helped all students increase their mathematical proficiency. While students did not work on tasks like this one, they did calculate angles of many different types of lines within 1-D, 2-D and 3-D. While all students demonstrated a significant difference, the researchers feel it is once again important to emphasize that the Coded LD students' performance on this task question shows that they can reach high levels of mathematical proficiency.

5. What changes occurred in overall achievement after administration of the UDL intervention?

   Table 11 below shows how the students performed as a total on pretest and posttest.

Table 12: Mean Scores For All Tasks

	N	Mean	Std. Deviation	Std. Error Mean
LD Coded Pretest	7	5.00	1.63	0.62
LD Coded Posttest	7	9.57	1.13	0.43
Not-Coded LD Pretest	20	6.25	2.75	0.62
Not-Coded LD Posttest	20	9.90	1.89	0.42
All Students Pretest	27	5.93	2.54	0.49
All Students Posttest	27	9.81	1.71	0.33

The graph below (Figure 17) shows an increase the overall mean scores between pretests and posttests on all items combined.

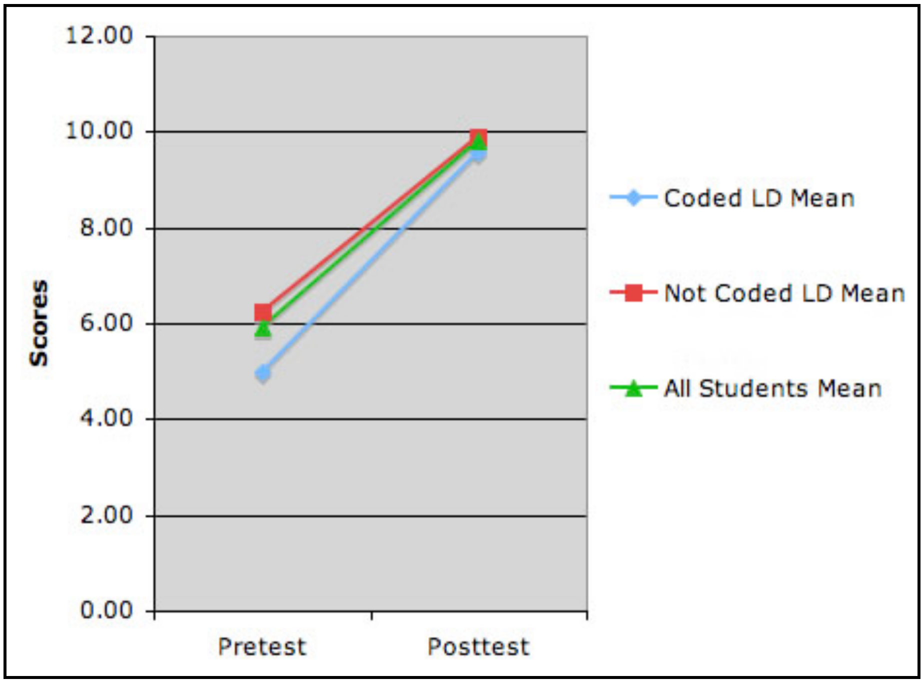

Figure 17: Mean Scores for Pretest Posttest for Overall Performance

Table 12 below indicates that the mean of overall scores is significantly different for Coded LD students, for Not Coded LD students and for all participating students. This represents a significant improvement in achievement for the Coded LD students and the Not Coded LD students. Overall, the class demonstrated significant improvement.

Table 13: Paired Samples *t*-test for Overall Achievement

Paired-Sample *t*-test	*t*	*df*	Sig. (2 tailed)	Mean Difference
Coded LD	-10.67	6	0.00	-4.57
Not Coded LD	-5.84	19	0.00	-3.65
All students	-8.13	26	0.00	-3.89

All students achieved a high level of mathematical proficiency as measured by these four PISA test items after the UDL intervention. This leads the researchers to conclude that designing mathematical learning for students as outlined in this study, leads to significantly increased student achievement.

Findings

Seven findings emerged from a close analysis of the data, which it is hoped will provide guidance to teachers, school jurisdiction leaders, policy makers and subsequent researchers as they consider:

- Employing UDL principles to build mathematical proficiency for *all* students
- Creating a curriculum that is accessible to *all* students
- Improving the achievement of special needs students
- Building the capacity of teachers to change mathematics teaching practices.

These findings are:

1. All students showed significant improvement in achievement
2. All students made gains in the five strands of mathematical proficiency.
3. All students can engage with difficult mathematical ideas when they are provided with dynamic assessment.
4. The principles of UDL permit teachers to break the stranglehold of the procedural script for teaching mathematics.
5. Access to technology is a critical factor in an accessible mathematics classroom
6. Introducing UDL into the mathematics classroom is a disruptive innovation.
7. Creating accessible mathematics classrooms, consistent with UDL principles, requires increased teacher knowledge and support for on-going professional development.

1. All students showed significant improvement in achievement.

The PISA test items used for pre and post-tests were chosen (1) for their validity and reliability and (2) for their ability to measure mathematical proficiency. The four items had levels of difficulty from middle to highest range. The instructional intervention was not designed to "teach to the test". Rather, all elements were designed to build mathematical proficiency that would transfer to a number of contexts, one of which is standardized testing of the highest international caliber.

Common sense worries about changing mathematics instruction to better meet the needs of special needs students were not realized. All students improved on all items. Mean scores for all tasks demonstrate statistically significant improvement for coded LD students, for not-coded LD students and for the class as a whole. Thus, it is possible to raise both the ceiling and the floor of student achievement by incorporating UDL principles into the design of mathematics curricula.

2. **All students demonstrated gains in the five strands of mathematical proficiency.**
   Kilpatrick, Swafford and Findell (2001) define mathematical proficiency in terms of five intertwining strands:
   - *conceptual understanding* – an understanding of concepts, operations and relations. Conceptual understanding frequently results in students' comprehending connections and similarities among interrelated facts.
   - *procedural fluency* – flexibility, accuracy and efficiency in implementing appropriate procedures. Skill in proficiency includes the knowledge of when and how to use procedures. This includes efficiency and accuracy in basic computations.
   - *strategic competence* – the ability to formulate, represent and solve mathematical problems. This is similar to problem solving. Strategic competence, conceptual understanding and procedural fluency are mutually supportive.
   - *adaptive reasoning* - the capacity to think logically about concepts and conceptual relationships. Reasoning is needed to navigate through the various procedures, facts and concepts required to arrive at solutions.
   - *productive disposition* – positive perceptions about mathematics. Productive disposition develops as students gain more mathematical understanding and become capable of learning and doing mathematics.

   Analysis of the qualitative data demonstrates the developing mathematical proficiency of students in this Grade 7 classroom as evidenced in their ability to dialogue with each other, to explore concepts in depth, to think and reason, to test conjectures and justify solutions.

   When considering the power of UDL principles to change the dominant procedural script of mathematics teaching, it is especially important to note that the instructional intervention involved five essential and connected elements: (1) mathematical content knowledge; (2) pedagogical content knowledge for mathematics; (3) UDL principles; (4) assessment for learning and (5) an instructional design process that supports the effective integration of mathematics strands as identified in the Program of Studies.

3. **All students can engage with difficult mathematical ideas when they are provided with dynamic assessment.**
   Assessment for learning places teachers and students in a design environment in which constant feedback informs next teaching and learning steps. As Black (2004) indicates, there is a great deal of confusion about the kinds of assessment that builds proficiency and improves achievement. In this study, students received dynamic feedback in a number of ways:
   - From teachers, in response to their individual work
   - From teachers, in response to the emergent design of lessons and activities to address misconceptions
   - From peers as they worked and talked together
   - From the learning environment, particularly in the case of the dynamic geometry software

   It is important to emphasize the difference between dynamic assessment and feedback through tests, quizzes and assignments designed for purposes of accountability, ranking of students, or certifying competence. The latter assessment practices are particularly damaging to students “with low attainments who are led to believe they lack ‘ability’ and are not able to learn”(Black, 2004, p.1).

   Learning goals remained the same for all students throughout the study. What changed was instructional design that included multiple means of representation and expression. When (1) the learning task was mathematically robust; (2) the representation of concepts was varied in pedagogically sound ways and (3) students were given a range of opportunities to express their emerging understandings, then all students were able to engage deeply. They volunteered their attention to and interest in the learning task.

4. **The principles of UDL permit teachers to break the stranglehold of the procedural script for teaching mathematics.**
   Creating more robust and interesting mathematical tasks, problems or inquiries is a necessary component of the design for accessible classrooms. However, it is not sufficient to provide more robust, complex problems intended to create mathematical proficiency (Stigler and Hiebert, 1999; Shanker Institute, 2005). The dominant North American script for teaching mathematics is so ingrained that teachers turned even the best problems into routine, procedural exercises.

Incorporating UDL principles into instructional design has the potential to change instruction at its root, disrupting the processes by which many students come to be labeled as unable to learn mathematics.

5. **Access to technology is a critical factor in an accessible mathematics classroom.**
Currently, the use of technology in UDL emphasizes the role of assistive technologies that permit students with identified needs to adapt to the pervasively print environment of most classrooms. AT has a definite role to play in creating more accessible learning opportunities for all students.

However, AT alone may leave untouched the procedural script for teaching mathematics if it leaves assumptions about the effective development of mathematical proficiency unchallenged. We can easily imagine classrooms in which, for examples, technology is introduced so that weaker students can in some sense keep up with the demands of fast-right-answer-giving, or where modifications that "dumb down" or fragment experiences are provided in the name of assistance.

What this study demonstrates is that the inherent nature of digital environments such as Geometer's Sketchpad™ and IO to represent and express mathematical concepts in dynamic ways.

6. **Introducing UDL into the mathematics classroom is a disruptive innovation.**
While the goal of creating increasingly accessible classrooms seems incontrovertible, actually creating the changes that make a difference for students disrupts the status quo.

(1) The research team made a number of other attempts to introduce range of technology-rich environments: email, wiki and access to a common drive and print-outs of emerging work. Difficulties with district policies and the school network and resources limited our opportunities to do this. As a work-around, we created an online website so that students could access Geometer's Sketchpad™ activities and instructions.

The school jurisdiction had made considerable effort to respond to findings of Friesen's 2006 report on accessible classrooms. On short notice, software was installed, enabled email and ensured that Universal Access features were accessible from all desktops. We appreciated these efforts, and could not have conducted the study without them. Nor could we have designed instruction incorporating principles of UDL had students not had access to laptop computers that functioned well.

In this report we note areas in which improvement is still possible.
(2) Rigid timetables, a ubiquitous feature of all secondary schools, interfere with the capacity of students to engage with learning in ways that build mathematical proficiency. When daily work is fragmented into short blocks of time, students and teachers become frustrated by arbitrary (and in our view, unnecessary) constraints on engagement.

Block timetables have a checkered history in secondary school reform. Unsuccessful attempts to introduce reform by increasing class times to 90 or 120 minute blocks without changing the teaching script, itself, have left both students and teachers frustrated. Doing more of the same kinds of procedural exercises, now for double or triple the time becomes excruciating. As with other elements in this study, we emphasize that a structural change, alone, will not make the kinds of difference we report here.

However, the teacher and student responses to constraints of the timetable confirm what we have found in other contexts. When students become engaged in the ways described here, both they and their teachers demand longer blocks of uninterrupted time for their work. It is our experience that introducing this kind of innovation inevitably puts pressure on existing structures such as the timetable.

(3) We had not anticipated the extent to which the increased proficiency of coded LD students disrupted the social hierarchies of the classroom. Students who considered themselves (or were considered by others) to be better at math were initially very uncomfortable with the emerging confidence and ability of students they thought were less able.

Disruptions of this sort point, perhaps, to the tenacity of conventional teaching scripts. When teachers and students experience initial discomfort at the introduction of innovation, it is tempting to retreat to familiar ground.

It is easy to pen the words that describe access for all to high levels of mathematical proficiency. It will be more challenging to live with the inevitable pressures that such a goal will place on taken-for-granted, everyday structures and experiences.

7. **Creating accessible mathematics classrooms consistent with UDL principles requires increased teacher knowledge and support for on-going professional development.**
   Changing teaching practices and school, jurisdiction and classroom structures will require significant investment in professional development.

(1) Most teachers, principals and senior administrators recognize the experience described by Mrs. Jamieson. They, themselves, have had unfortunate experiences with math in school—or they know many people in the same boat. Teaching mathematics that incorporates UDL principles requires teachers to design learning experiences in mathematics that they, themselves, have never experienced.

Progress will require the active engagement of mathematicians and math educators to design pedagogical content knowledge that is mathematically sound. More math courses of the procedural sort will not get teachers out of their current dilemma. While most need more mathematics, it is mathematics of a particular sort: the kind that permits them to design instruction that gives students access to complex ideas.

Mrs. Jamieson reported to the research team that she had followed up her involvement in the study with a summer course in mathematics. Knowing that she, herself, needed a deeper understanding of mathematical concepts, she was also clear that a course, alone, was unlikely to help in the ways she desired. “I want to be able to think like you,” she told Dr. Friesen. And to do that, she knew she wanted more opportunities to explore pedagogical issues at the same time.

(2) Leadership to support teachers like Mrs. Jamieson, to provide them with useful feedback on their teaching for professional growth, and to make sound judgments about administrative issues such as timetables and allocation of resources requires a degree of understanding of UDL principles and mathematics that principals generally lack at this time. Developing the capacity to lead for learning of this sort cannot be left to chance.

Recommendations

1. Create a curriculum for mathematics that draws upon the principles of UDL.

Context

Mathematics has several key elements for curriculum design using UDL strongly in place:

- Organizations such as the Pacific Institute for Mathematical Sciences (PIMS) have already indicated a strong desire to bring mathematicians, math educators and teachers to together to create robust problems and instructional design that increases math proficiency in both teachers and students.

- PIMS has already created a network of mathematicians, math educators, teachers and First Nations and Metis Elders to address the particular concerns of mathematics and First Nations and Metis students. This demonstrates PIMS willingness and capacity to address the issue of making mathematics accessible to all.
- The National Science Foundation has invested heavily in on-going work to create mathematically robust and engaging problems available at no cost to teachers.
- There is a developing history of professional development through Lesson Study in Canada and the US which involves mathematicians, math educators and teachers.
- The International Assessment Consortium from UK continue to identify the problems in practice with assessment—particularly struggles teachers have to build assessment for learning into their practice. We can build upon and contribute to this work.
- Alberta Education has a strong interest in exploring the application of UDL in general and in mathematics in particular.

That is, key elements of designing effectively for UDL in the mathematics classroom are already in place in other contexts. Alberta Education could draw quickly upon these resources to create an Alberta-made approach to the creation of accessible classrooms in mathematics.

The successful creation of this Alberta solution to the problem of raising the ceiling and lifting the floor could provide a model for changes to all subject areas.

Implications for Alberta Education

- Look for synergy partners like PIMS, the International Assessment Consortium and CAST who understand the particular issues of teaching mathematics, assessment for learning and UDL.
- Special Programs Branch should take the lead in bringing partners together to create a mathematics curriculum (understood in its broadest sense) designed according to the principles of UDL.
- Special Programs Branch should take the lead in developing and publishing resources that represent both mathematics and pedagogical content in multiple, flexible and technologically sophisticated ways.

Implications for Universities

- PIMS mathematicians and math educators come from the universities. Their involvement in creating a provincial curriculum is essential. It is also hoped that their involvement in this project would increase the effectiveness of teacher preparation in mathematics.

- Faculties of education must address the development of proficiency in all students, not as a special education topic, but as an integral part of their curriculum and instruction courses.

2. **Establish a network of teachers who are willing to form a Community of Practice.**

Context
Conventionally, new curricula are developed by some and delivered by others. In the U.S. we have seen the failure of this approach, even to the creation and dissemination of mathematically robust problems. Recommendation #2 suggests that the development of a mathematics curriculum based on UDL will require design research in which teachers are involved from the outset in multiple ways: in dialogue with mathematicians and math educators; in working through robust problems to increase their own mathematical understandings; to dialogue as they work in their classrooms; and to make their practice public so that others in the network can build their own mathematical and pedagogical proficiency. In essence, we are suggesting a new approach to developing curriculum by prototyping the innovation as it is being created.

In this report, we have suggested the potential pitfalls of attaching UDL principles to tenacious procedural scripts for the teaching of mathematics. It is easy to read about such principles and quickly assume that one knows how to teach with them. We anticipate, for example, educators who will dismiss their power by saying, "They are just good practice. There's nothing really new in all this."

If that happens, then the province will suffer a rash of "multiples" stuck on to existing resources and procedures. We do not underestimate the danger of this, nor the care with which one must proceed to develop innovations that will actually take hold effectively.

The support and active involvement of teachers willing to do what Mrs. Jamieson did—to try unfamiliar approaches over an extended period of time—will be key to the innovation's success.

Alberta has the technological broadband infrastructure through SuperNet to permit teachers to connect in both synchronous and asynchronous ways. The community of practice does not need to be geographically limited. In fact, in terms of addressing issues of diversity, the capacity to have teachers from across the province--rural, urban, First Nations--working on the same issues is essential.

Implications for Alberta Education

- Special Programs Branch should take the organizing lead in bringing this network together and supporting its work with funding and resources.
- Special Programs Branch should issue a request for proposals (RFP) to school jurisdictions to become part of this network. This will ensure that school jurisdictions get behind the initiative.
- Design the RFP to include stipulations for buy-out time for participating teachers. In the past, CANARIE-sponsored initiatives provided participating teachers with a day a week to devote appropriate time and attention.
- Establish a research committee to conduct design research on the work and outcomes of the network

Implications for School Jurisdictions

- Allocate resources to the initiative.
- Develop processes to feed emerging work from the network back into the jurisdiction to develop the capacity of others to work in these ways
- Provide and support the necessary technological infrastructure

Implications for Principals

- Develop the instructional leadership capacity to direct and supervise work at the school level. Few principals will have taught in these ways, and it cannot be assumed that they will be able to give the most useful feedback possible when teachers introduce the innovation in their classrooms. It would do a disservice to principals and to teachers to establish a myth that UDL principles are just like all the other good things they have always done. Leaders must understand and be able to act on the differences that make a difference.
- Disruptions to the status quo are bound to occur. Of necessity, for example,
    - the need for new timetables may emerge.
    - understanding the dynamics of anticipated and unanticipated resistance that puts pressure on teachers to revert to conventional practices.

Implications for Teachers

- Active participation in a design research network will take time for participating teachers. It is unreasonable to ask people to do pioneering work without providing additional time and resources they find meaningful.
- Participants will be asked to demonstrate willingness to:
    - increase their own mathematical proficiency

- learn the principles of UDL and understand their application to mathematics in particular
- use technology both to represent concepts to students and to permit students to express knowledge in multiple ways
- collaborate with others in ways that build new knowledge and “next practice”
- make their practice increasingly public by sharing video clips; student responses to the work; struggles and successes in developing next practices, etc.

References

Adams, R., & Wu, M. (Eds.). (2002). *PISA 2000 technical report*. Paris: Organization for Economic Co-operation and Development (OECD). Retrieved January 7, 2008 from http://www.pisa.oecd.org/dataoecd/53/19/33688233.pdf

Alberta Education (2006). Individualized program planning. Retrieved January 7, 2008 from http://www.education.alberta.ca/media/511715/ipp.pdf

Alberta Education (2007). Mathematics kindergarten to grade 9 program of studies. Alberta Education. Retrieved January 7, 2008 from http://education.alberta.ca/media/645594/kto9math.pdf

Assessment Reform Group (1999). Assessment for learning: Beyond the black box. Cambridge: UK: Cambridge School of Education. Retrieved January 7, 2008 from http://k1.ioe.ac.uk/tlrp/arg/AssessInsides.pdf

Ball, D. & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis & E. Simmt (Eds.), *Proceedings of the 2002 Annual Meeting of the Canadian Mathematics Education Study Group,* (pp.3-14). Edmonton, AB: CMESG/GCEDM.

Barab, S. & Squire, K. (2004). Design-based research: Putting a stake in the ground. *The Journal of the Learning Sciences*, *13*(1), 1-14.

Bausch, M & Hasselbring, T. (2005). Using AT: Is it working? Eye on Research. Threshold. Retrieved January 7, 2008 from http://www.ciconline.org/threshold.

Bereiter, C. (2002). Design research for sustained innovation. *Cognitive studies, Bulletin of the Japanese Cognitive Science Society, 9*(3), 321–327.

Black, P. (2004). The nature and value of formative assessment for learning. Assessment for Learning Group. Retrieved January 7, 2008 from http://www.kcl.ac.uk/content/1/c4/73/57/formative.pdf

Black, P. & Wiliam, D (1998a). Assessment and Classroom Learning. *Assessment in Education*, 5(1), 7-71.

Black, P. J.; Harrison, C.; Lee, C.; Marshall, B. & Wiliam, D (2002). *Working inside the black box: Assessment for learning in the classroom*. London, UK: King's College London School of Education.

Bransford, J., Brown, A. & Cocking, R. (eds) (2000). *How people learn: Brain, mind, experience and school.* Washington, DC: National Academies Press.

Cardinali, R., & Gordon, Z. (2002). Technology: Making things easier for all of us - for the disabled making things possible. *Equal Opportunities International*, 21(1): 65-79.

CAST (2006). Retrieved January 7, 2008 from http://www.cast.org

Clifford, P. & Friesen, S. (2001). Bringing Learning to Learners: The Galileo Educational Network. Paper presented at *ED-MEDIA 2001: World Conference on Educational Multimedia, Hypermedia and Telecommunicat*ions, Tampere, Finland.

Cohen, R. B. (2005). Examining the work of constructing a representational context in elementary mathematics teaching. Unpublished doctoral dissertation, University of Michigan, Ann Arbor, MI.

Davies, A. (2002-2003). *Finding proof of learning in a one-to-one computing classroom*. Courtenay, B.C.: Connections Publishing.

Dolan, R. P. & Hall, T. E. (2001). Universal Design for Learning: Implications for Large-Scale Assessment. *IDA Perspectives* 27(4): 22-25.

Donovan, M.S. & Bransford, J. (2005). Pulling threads. In M.S. Donovan & J. Bransford (Eds.), *How students learn: Mathematics in the classroom.* (pp. 569-590). Washington, DC: National Academies Press.

Edyburn, D. (2005). A primer on Universal Design (UD) in education. Retrieved January 7, 2008 from http://www.uwm.edu/~edyburn/ud.html.

Edyburn, D. (2006a). Failure is not an option: Collecting, reviewing, and acting on
evidence for using technology to enhance academic performance. *Learning and Leading With Technology*, 34(1), 20-23.

Edyburn, D. (2006b). Principles of universal design and the implications for policy initiatives. Unpublished manuscript.

Edyburn, D. (2006c). Re-examining the role of assistive technology in learning. Closing the Gap. 25(5), 10-11, 26. Retrieved January 7, 2008 from http://www.uwm.edu/%7eedyburn/ATinDepth.pdf

Firchow, N. (2002) Universal Design for Learning: Improved access for all. Retrieved January 7, 2008 from SchwabLearning.org http://www.schwablearning.org/articles.aspx?r=490

Friesen, S. (2007). Inside an accessible classroom. Unpublished manuscript.

Fuson, K., Kalchman, M. & Bransford, J. (2005). Mathematical understanding: An introduction. In M.S. Donovan & J. Bransford (Eds.), *How students learn: Mathematics in the classroom.* (pp. 217-256). Washington, DC: National Academies Press.

Gay, L., Mills, G., & Airasian, P. (2006). *Educational research: Competencies for analysis and applications* (8th ed.). New Jersey: Pearson.

Gilbert, J. (2005). *Catching the knowledge wave? The knowledge society and the future of education.* Wellington, NZ: NZCER Press.

Hendricks, P. & Daley, J. (2004). Technology: Is It Accessible to All Students? Mid-Atlantic Regional Technology in Education Consortium. Retrieved January 7, 2008, from: http://www.temple.edu/MARTEC/publications/technobriefs/tbrief39.html

International Association for the Evaluation of Education Achievement (1995). Third international mathematics and science study. Boston College. Retrieved December 30, 2007 from http://timss.bc.edu/

Kaplan, R. & Kaplan, E. (2007). *Out of the labyrinth: Setting mathematics free*. New York: Oxford University Press.

Kelly, A. (2003). Theme issue: The role of design in educational research. *Educational Researcher, 32*(1), 3–4.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.) (2001). *Adding it up: Helping children learn mathematics*. Washington, D.C.: National Academy Press.

Leininger, M. M. (1985). Ethnography and ethnonursing: Models and modes of qualitative data analysis. In M. M. Leininger (Ed.), *Qualitative research methods in nursing* (pp. 33-72). Orlando, FL: Grune & Stratton.

Meo, G. (2005). Curriculum access for all: Universal Design for Learning. *Harvard Education Letter,* 21(5). Retrieved January 7, 2008 from http://www.edletter.org/past/issues/2005-nd/meo.shtml

Mighton, J. (2007). The end of ignorance: Multiplying our human potential. Toronto, ON: Knopf Canada.

Mighton, J. (2003). *The myth of ability: Nurturing mathematical talent in every child.* Toronto, ON: House of Anansi Press Inc.

Milgram, J. (2005). *The mathematics pre-service teachers need to know.* Retrieved January 7, 2008 from http://hub.mspnet.org/exit.cfm/FIE_book.pdf?url=ftp%3A%2F%2Fmath%2Estanford%2Eedu%2Fpub%2Fpapers%2Fmilgram%2FFIE%2Dbook%2Epdf.

NCTM (2000). Executive summary: Principles and standards for school mathematics. NCTM. Retrieved January 7, 2008 from http://www.nctm.org%2FuploadedFiles%2FMath_Standards%2F12752_exec_pssm.pdf&ei=hAx9R53OOaSkgQLmnoQ1&usg=AFQjCNHrgv1d4nw8aJTT3Wc0TA2kB0oKDw&sig2=rZ6KDzPigEXABUXH2RDqtA

Peters, S. (2007). Education for all? *Journal of Disability Policy Studies*. 18(2), (Fall), p.98-108.

Pittard, V. (2004). Evidence for e-learning policy. *Technology, Pedagogy and Education*, 13(2), pp. 181–194.

Research Points (2006). Do the math: Cognitive demand makes a difference. *Research Points*, 4(2). AERA.

Rose, D. & Meyer, A. (2002). Teaching every student in the digital age: Universal design for learning. ASCD. Retrieved January 7, 2008 from http://www.cast.org/teachingeverystudent/ideas/tes/.

Rose, D.H., & Meyer, A. (2006). *A practical reader in Universal design for learning.* Cambridge, MA: Harvard Education Press.

Rose, D., Meyer, A. & Hitchcock, C. (Eds). (2005). *The universally designed classroom: Accessible curriculum and digital technologies*. Cambridge, MA: Harvard University Press.

Rothberg, M & Treviranus, J. (2006). Accessible e-learning demonstrations using IMS accessibility specifications. ATIA National Conference, Orlando, Florida. Retrieved January 7, 2008 from http://ncdae.org/activities/atia06/presentations.cfm

Senechal, M. (1990). Shape. In L. Steen (Ed.) On the shoulders of giants: New approaches to numeracy. Washington, D.C.: National Academies Press.

Shanker Institute (2005, May). From Best Research to What Works: Improving the Teaching and Learning of Mathematics: A Forum (Washington, D. C., May 5, 2005).

Stiggins, R., Arter, J.; Chappuis, J. & Chappius, S. (2004). *Classroom assessment for student achievement: Doing it right – using it well.* Portland, Oregon: Assessment Training Institute.

Stigler, J. & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom.* New York, NY: The Free Press.

Swain, J. & Swan, M. (2007). Thinking through mathematics: Research report. NRDC. Retrieved January 7, 2008 from http://www.maths4life.org/uploads/documents/doc_296.pdf.

The Alberta Teachers Association (ATA). (2003). Trying to teaching, trying to learn: Listening to students. Edmonton, AB: The Alberta Teachers Association.

The Design-Based Research Collective. (2003). Design-based research: An emerging paradigm for educational inquiry. *Educational Researcher*, 32(1), 5-8.

U.S. Department of Education. (2004, March 16). Professional development and innovative tools for learning science . Panel 2 discussion at the Secretary's Summit on Science, Washington, DC.

van den Akker, J., Graveemeijer, K., McKenney, S. and Nieveen, N. (eds). (2006). *Educational Design Research*. London and New York: Routledge.

Wineburg, S., & Wilson, S. M. (1991). Subject-matter knowledge in the teaching of history. In J. Brophy (Ed.), Advances in research on teaching (Vol. 2, pp. 305-345). Greenwich, CT: JAI Press.