

Big Mathematical Ideas and Understandings

A **Big Idea** is a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole.

BIG IDEA #1

NUMBERS — The set of real numbers is infinite, and each real number can be associated with a unique point on the number line.

Examples of Mathematical Understandings:

Counting Numbers

- Counting tells how many items there are altogether. When counting, the last number tells the total number of items; it is a cumulative count.
- Counting a set in a different order does not change the total.
- There is a number word and a matching symbol that tell exactly how many items are in a group.
- Each counting number can be associated with a unique point on the number line, but there are many points on the number line that cannot be named by the counting numbers.
- The distance between any two consecutive counting numbers on a given number line is the same.
- One is the least counting number and there is no greatest counting number on the number line.
- Numbers can also be used to tell the position of objects in a sequence (e.g., 3rd), and numbers can be used to name something (e.g., social security numbers).

Whole Numbers

- Zero is a number used to describe how many are in a group with no objects in it.
- Zero can be associated with a unique point on the number line.
- Each whole number can be associated with a unique point on the number line, but there are many points on the number line that cannot be named by the whole numbers.
- Zero is the least whole number and there is no greatest whole number on the number line.
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Integers

- Integers are the whole numbers and their opposites on the number line, where zero is its own opposite.

Adapted from Charles, R. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. Journal of Mathematics Education Leadership, 7(3), 9-21.

- Each integer can be associated with a unique point on the number line, but there are many points on the number line that cannot be named by integers.
- An integer and its opposite are the same distance from zero on the number line.
- There is no greatest or least integer on the number line.

Fractions/Rational Numbers

- A fraction describes the division of a whole (region, set, segment) into equal parts.
- The bottom number in a fraction tells how many equal parts the whole or unit is divided into. The top number tells how many equal parts are indicated.
- A fraction is relative to the size of the whole or unit.
- A fraction describes division. ($a/b = a \div b$, a & b are integers & $b \neq 0$), and it can be interpreted on the number line in two ways. For example, $2/3 = 2 \div 3$. On the number line, $2 \div 3$ can be interpreted as 2 segments where each is $1/3$ of a unit ($2 \times 1/3$) or $1/3$ of 2 whole units ($1/3 \times 2$); each is associated with the same point on the number line. (Rational number)
- Each fraction can be associated with a unique point on the number line, but not all of the points between integers can be named by fractions.
- There is no least or greatest fraction on the number line.
- There are an infinite number of fractions between any two fractions on the number line.
- A decimal is another name for a fraction and thus can be associated with the corresponding point on the number line.
- Whole numbers and integers can be written as fractions (e.g., $4 = 4/1$, $-2 = -2/1$).
- A percent is another way to write a decimal that compares part to a whole where the whole is 100 and thus can be associated with the corresponding point on the number line.
- Percent is relative to the size of the whole.

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BIG IDEA #2

THE BASE TEN NUMERATION SYSTEM — The base ten numeration system is a scheme for recording numbers using digits 0-9, groups of ten, and place value.

Examples of Mathematical Understandings:

Whole Numbers

- Numbers can be represented using objects, words, and symbols.
- For any number, the place of a digit tells how many ones, tens, hundreds, and so forth are represented by that digit.
- Each place value to the left of another is ten times greater than the one to the right (e.g., $100 = 10 \times 10$).
- You can add the value of the digits together to get the value of the number.
- Sets of ten, one hundred and so forth must be perceived as single entities when interpreting numbers using place value (e.g., 1 hundred is one group, it is 10 tens or 100 ones).

Decimals

- Decimal place value is an extension of whole number place value.
- The base-ten numeration system extends infinitely to very large and very small numbers (e.g., millions & millionths).

BIG IDEA #3

EQUIVALENCE: Any number, measure, numerical expression, algebraic expression, or equation can be represented in an infinite number of ways that have the same value.

Examples of Mathematical Understandings:

Numbers and Numeration

- Numbers can be decomposed into parts in an infinite number of ways
- Numbers can be named in equivalent ways using place value (e.g., 2 hundreds 4 tens is equivalent to 24 tens).
- Numerical expressions can be named in an infinite number of different but equivalent ways (e.g., $4/6 \div 2/8 = 2/3 \div 1/4 = 2/3 \times 4/1$; also $26 \times 4 = (20 + 6) \times 4$).
- Decimal numbers can be named in an infinite number of different but equivalent forms (e.g., $0.3 = 0.30 = 0.10 + 0.20$).

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Number Theory and Fractions

- Every composite number can be expressed as the product of prime numbers in exactly one way, disregarding the order of the factors (Fundamental Theorem of Arithmetic).
- Every fraction/ratio can be represented by an infinite set of different but equivalent fractions/ratios.

Algebraic Expressions and Equations

- Algebraic expressions can be named in an infinite number of different but equivalent ways (e.g., $2(x - 12) = 2x - 24 = 2x - (28 - 4)$).
- A given equation can be represented in an infinite number of different ways that have the same solution (e.g., $3x - 5 = 16$ and $3x = 21$ are equivalent equations; they have the same solution, 7).

Measurement

- Measurements can be represented in equivalent ways using different units (e.g., 2 ft 3 in = 27 in.).
- A given time of day can be represented in more than one way.
- For most money amounts, there are different, but finite combinations of currency that show the same amount; the number of coins in two sets does not necessarily indicate which of two sets has the greater value.

BIG IDEA #4

COMPARISON: Numbers, expressions, and measures can be compared by their relative values.

Examples of Mathematical Understandings:

Numbers & Expressions

- One-to-one correspondence can be used to compare sets.
- A number to the right of another on the number line is the greater number.
- Numbers can be compared using greater than, less than, or equal.
- Three or more numbers can be ordered by repeatedly doing pair-wise comparisons.
- Whole numbers and decimals can be compared by analyzing corresponding place values.
- Numerical and algebraic expressions can be compared using greater than, less than, or equal.

Adapted from Charles, R. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. Journal of Mathematics Education Leadership, 7(3), 9-21.

Fractions, Ratios, & Percent

- A comparison of a part to the whole can be represented using a fraction.
- A ratio is a multiplicative comparison of quantities; there are different types of comparisons that can be represented as ratios.
- Ratios give the relative sizes of the quantities being compared, not necessarily the actual sizes.
- Rates are special types of ratios where unlike quantities are being compared.
- A percent is a special type of ratio where a part is compared to a whole and the whole is 100.
- The probability of an event is a special type of ratio.

Geometry and Measurement

- Lengths can be compared using ideas such as longer, shorter, and equal.
- Mass/weights can be compared using ideas such as heavier, lighter, and equal.
- Measures of area, volume, capacity and temperature can each be compared using ideas such as greater than, less than, and equal.
- Time duration for events can be compared using ideas such as longer, shorter, and equal.
- Angles can be compared using ideas such as greater than, less than, and equal.

BIG IDEA #5

OPERATION MEANINGS & RELATIONSHIPS: The same number sentence (e.g. $12-4 = 8$) can be associated with different concrete or real-world situations, AND different number sentences can be associated with the same concrete or real-world situation.

Examples of Mathematical Understandings:

Whole Numbers

- Some real-world problems involving joining, separating, part-part-whole, or comparison can be solved using addition; others can be solved using subtraction.
- Adding x is the inverse of subtracting x .

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- Any subtraction calculation can be solved by adding up from the subtrahend.
- Adding quantities greater than zero gives a sum that's greater than any addend.
- Subtracting a whole number (except 0) from another whole number gives a difference that's less than the minuend.
- Some real-world problems involving joining equal groups, separating equal groups, comparison, or combinations can be solved using multiplication; others can be solved using division.
- Multiplying by x is the inverse of dividing by x .
- Any division calculation can be solved using multiplication.
- Multiplying two whole numbers greater than one gives a product greater than either factor.

Rational Numbers (Fractions & Decimals)

- The real-world actions for addition and subtraction of whole numbers are the same for operations with fractions and decimals.
- Different real-world interpretations can be associated with the product of a whole number and fraction (decimal), a fraction (decimal) and whole number, and a fraction and fraction (decimal and decimal).
- Different real-world interpretations can be associated with division calculations involving fractions (decimals).
- The effects of operations for addition and subtraction with fractions and decimals are the same as those with whole numbers.
- The product of two positive fractions each less than one is less than either factor.

Integers

- The real-world actions for operations with integers are the same for operations with whole numbers.

BIG IDEA #6

PROPERTIES: For a given set of numbers there are relationships that are always true, and these are the rules that govern arithmetic and algebra.

Examples of Mathematical Understandings:

Properties of Operations

Adapted from Charles, R. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. Journal of Mathematics Education Leadership, 7(3), 9-21.

- Properties of whole numbers apply to certain operations but not others (e.g. The commutative property applies to addition and multiplication but not subtraction and division.).
- Two numbers can be added in any order; two numbers can be multiplied in any order.
- The sum of a number and zero is the number; the product of any non-zero number and one is the number.
- Three or more numbers can be grouped and added (or multiplied) in any order.

Properties of Equality

- If the same real number is added or subtracted to both sides of an equation, equality is maintained.
- If both sides of an equation are multiplied or divided by the same real number (not dividing by 0), equality is maintained.
- Two quantities equal to the same third quantity are equal to each other.

BIG IDEA #7

BASIC FACTS & ALGORITHMS: Basic facts and algorithms for operations with rational numbers use notions of equivalence to transform calculations into simpler ones.

Examples of Mathematical Understandings:

Mental Calculations

- Number relationships and sequences can be used for mental calculations (one more, one less; ten more, ten less; 30 is two more than 28; counting back by thousands from 50,000 is 49,000, 48,000, 47,000 etc.)
- Numbers can be broken apart and grouped in different ways to make calculations simpler.

Whole Number Basic Facts & Algorithms

- Some basic addition and multiplication facts can be found by breaking apart the unknown fact into known facts. Then the answers to the known facts are combined to give the final value.
- Subtraction facts can be found by thinking of the related addition fact.
- Division facts can be found by thinking about the related multiplication fact.
- When 0 is divided by any non-zero number, the quotient is zero, and 0 cannot be a divisor.
- Addition can be used to check subtraction, and multiplication can be used to check division.

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- Powers of ten are important benchmarks in our numeration system, and thinking about numbers in relation to powers of ten can make addition and subtraction easier.
- When you divide whole numbers sometimes there is a remainder; the remainder must be less than the divisor.
- The real-world situation determines how a remainder needs to be interpreted when solving a problem.

Rational Number Algorithms

- Fractions with unlike denominators are renamed as equivalent fractions with like denominators to add and subtract.
- The product of two fractions can be found by multiplying numerators and multiplying denominators.
- A fraction division calculation can be changed to an equivalent multiplication calculation (i.e., $a/b \div c/d = a/b \times d/c$, where $b, c,$ and $d \neq 0$).
- Division with a decimal divisor is changed to an equivalent calculation with a whole number divisor by multiplying the
- divisor and dividend by an appropriate power of ten.
- Money amounts represented as decimals can be added and subtracted using the same algorithms as with whole numbers.

Measurement

- Algorithms for operations with measures are modifications of algorithms for rational numbers.
- Length measurements in feet and inches can be added or subtracted where 1 foot is regrouped as 12 inches.
- Times in minutes and seconds can be added and subtracted where 1 minute is regrouped as 60 seconds.

BIG IDEA #8

ESTIMATION: Numerical calculations can be approximated by replacing numbers with other numbers that are close and easy to compute with mentally. Measurements can be approximated using known referents as the unit in the measurement process.

Examples of Mathematical Understandings:

Numerical

- The numbers used to make an estimate determine whether the estimate is over or under the exact answer.
- Division algorithms use numerical estimation and the relationship between division and multiplication to find quotients.

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- Benchmark fractions like $\frac{1}{2}$ (0.5) and $\frac{1}{4}$ (0.25) can be used to estimate calculations involving fractions and decimals.
- Estimation can be used to check the reasonableness of exact answers found by paper/pencil or calculator methods.

Measurement

- Length, area, volume, and mass/weight measurements can be estimated using appropriate known referents.
- A large number of objects in a given area can be estimated by finding how many are in a sub-section and multiplying by the number of sub-sections.

Adapted from Charles, R. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. Journal of Mathematics Education Leadership, 7(3), 9-21.

BIG IDEA #9

PATTERNS: Relationships can be described and generalizations made for mathematical situations that have numbers or objects that repeat in predictable ways.

Examples of Mathematical Understandings:

Numbers

- Skip counting on the number line generates number patterns.
- The structure of the base ten numeration system produces many numerical patterns.
- There are patterns in the products for multiplication facts with factors of 0, 1, 2, 5, and 9.
- There are patterns when multiplying or dividing whole numbers and decimals by powers of ten.
- The difference between successive terms in some sequences is constant.
- The ratio of successive terms in some sequences is a constant.
- Known elements in a pattern can be used to predict other elements.

Geometry

- Some sequences of geometric objects change in predictable ways.

BIG IDEA #10

VARIABLE: Mathematical situations and structures can be translated and represented abstractly using variables, expressions, and equations.

Examples of Mathematical Understandings:

- Letters are used in mathematics to represent generalized properties, unknowns in equations, and relationships between quantities.
- Some mathematical phrases can be represented as algebraic expressions (e.g. Five less than a number can be written as $n - 5$.)
- Some problem situations can be represented as algebraic expressions (e.g. Susan is twice as tall as Tom; If $T = \text{Tom's height}$, then $2T = \text{Susan's height}$.)
- Algebraic expressions can be used to generalize some transformations of objects in the plane.

Adapted from Charles, R. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. Journal of Mathematics Education Leadership, 7(3), 9-21.

BIG IDEA #11

PROPORTIONALITY: If two quantities vary proportionally, that relationship can be represented as a linear function.

Examples of Mathematical Understandings:

- A ratio is a multiplicative comparison of quantities.
- Ratios give the relative sizes of the quantities being compared, not necessarily the actual sizes.
- Ratios can be expressed as units by finding an equivalent ratio where the second term is one.
- A proportion is a relationship between relationships.
- If two quantities vary proportionally, the ratio of corresponding terms is constant.
- If two quantities vary proportionally, the constant ratio can be expressed in lowest terms (a composite unit) or as a unit amount; the constant ratio is the slope of the related linear function.
- There are several techniques for solving proportions (e.g., finding the unit amount, cross products).
- When you graph the terms of equal ratios as ordered pairs (first term, second term) and connect the points, the graph is a straight line.
- If two quantities vary proportionally, the quantities are either directly related (as one increases the other increases) or inversely related (as one increases the other decreases).
- Scale drawings involve similar figures, and corresponding parts of similar figures are proportional.
- In any circle, the ratio of the circumference to the diameter is always the same and is represented by the number pi.
- Rates can be related using proportions as can percents and probabilities.

BIG IDEA #12

RELATIONS & FUNCTIONS: Mathematical rules (relations) can be used to assign members of one set to members of another set. A special rule (function) assigns each member of one set to a unique member of the other set.

Examples of Mathematical Understandings:

- Mathematical relationships can be represented and analyzed using words, tables, graphs, and equations.
- In mathematical relationships, the value for one quantity depends on the value of the other quantity.
- The nature of the quantities in a relationship determines what values of the input and output quantities are reasonable.

Adapted from Charles, R. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. Journal of Mathematics Education Leadership, 7(3), 9-21.

- The graph of a relationship can be analyzed with regard to the change in one quantity relative to the change in the other quantity.

The graph of a relation can be analyzed to determine if the relation is a function.

- In a linear function of the form $y = ax$, a is the constant of variation and it represents the rate of change of y with respect to x .
- The solutions to a linear function form a straight line when graphed.
- A horizontal line has a slope of 0, and a vertical line does not have a slope.
- The parameters in an equation representing a function affect the graph of the function in predictable ways.

BIG IDEA #13

EQUATIONS & INEQUALITIES: Rules of arithmetic and algebra can be used together with notions of equivalence to transform equations and inequalities so solutions can be found.

Examples of Mathematical Understandings:

- A solution to an equation is a value of the unknown or unknowns that makes the equation true.
- Properties of equality and reversible operations can be used to generate equivalent equations and find solutions.
- Techniques for solving equations start by transforming the equation into an equivalent one.
- A solution or solutions to a linear or quadratic equation can be found in the table of ordered pairs or from the graph of the related function.
- Techniques for solving equations can be applied to solving inequalities, but the direction of the inequality sign needs to be considered when negative numbers are involved.

BIG IDEA #14

SHAPES & SOLIDS: Two- and three-dimensional objects with or without curved surfaces can be described, classified, and analyzed by their attributes.

Examples of Mathematical Understandings:

- Point, line, line segment, and plane are the core attributes of space objects, and real-world situations can be used to think about these attributes.
- Polygons can be described uniquely by their sides and angles.
- Polygons can be constructed from or decomposed into other polygons.
- Triangles and quadrilaterals can be described, categorized, and named based on the relative lengths of their sides and the sizes of their angles.

Adapted from Charles, R. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. Journal of Mathematics Education Leadership, 7(3), 9-21.

- All polyhedra can be described completely by their faces, edges, and vertices.
- Some shapes or combinations of shapes can be put together without overlapping to completely cover the plane.
- • There is more than one way to classify most shapes and solids.

BIG IDEA #15

ORIENTATION & LOCATION: Objects in space can be oriented in an infinite number of ways, and an object's location in space can be described quantitatively.

Examples of Mathematical Understandings:

Lines and Line Segments

- Two distinct lines in the plane are either parallel or intersecting; two distinct lines in space are parallel, intersecting
- or skew.
- The angles formed by two intersecting lines in the plane are related in special ways (e.g., vertical angles).
- A number of degrees can be used to describe the size of an angle's opening.
- Some angles have special relationships based on their position or measures (e.g., complementary angles).
- In the plane, when a line intersects two parallel lines the angles formed are related in special ways.

Objects

- The orientation of an object does not change the other attributes of the object.
- The Cartesian Coordinate System is a scheme that uses two perpendicular number lines intersecting at 0 on each to
- name the location of points in the plane; the system can be extended to name points in space.
- Every point in the plane can be described uniquely by an ordered pair of numbers; the first number tells the distance to
- the left or right of zero on the horizontal number line; the second tells the distance above or below zero on the vertical
- number line.

BIG IDEA #16

Adapted from Charles, R. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. Journal of Mathematics Education Leadership, 7(3), 9-21.

TRANSFORMATIONS: Objects in space can be transformed in an infinite number of ways, and those transformations can be described and analyzed mathematically.

Examples of Mathematical Understandings:

- Congruent figures remain congruent through translations, rotations, and reflections.
- Shapes can be transformed to similar shapes (but larger or smaller) with proportional corresponding sides and congruent corresponding angles
- Algebraic expressions can be used to generalize transformations for objects in the plane.
- Some shapes can be divided in half where one half folds exactly on top of the other (line symmetry).
- Some shapes can be rotated around a point in less than one complete turn and land exactly on top of themselves (rotational symmetry).

BIG IDEA #17

MEASUREMENT: Some attributes of objects are measurable and can be quantified using unit amounts.

Examples of Mathematical Understandings:

- Measurement involves a selected attribute of an object (length, area, mass, volume, capacity) and a comparison of the object being measured against a unit of the same attribute.
- The longer the unit of measure, the fewer units it takes to measure the object.
- The magnitude of the attribute to be measured and the accuracy needed determines the appropriate measurement unit.
- For a given perimeter there can be a shape with area close to zero. The maximum area for a given perimeter and a given number of sides is the regular polygon with that number of sides

BIG IDEA #18

DATA COLLECTION: Some questions can be answered by collecting and analyzing data, and the question to be answered determines the data that needs to be collected and how best to collect it.

Examples of Mathematical Understandings:

- An appropriately selected sample can be used to describe and make predictions about a population.

Adapted from Charles, R. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. Journal of Mathematics Education Leadership, 7(3), 9-21.

- The size of a sample determines how close data from the sample mirrors the population.

Adapted from Charles, R. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. Journal of Mathematics Education Leadership, 7(3), 9-21.

BIG IDEA #19

DATA REPRESENTATION: Data can be represented visually using tables, charts, and graphs.

The type of data determines the best choice of visual representation.

Examples of Mathematical Understandings:

- Each type of graph is most appropriate for certain types of data.
- Scale influences the patterns that can be observed in data.

BIG IDEA #20

DATA DISTRIBUTION: There are special numerical measures that describe the center and spread of numerical data sets.

Examples of Mathematical Understandings:

- The best descriptor of the center of a numerical data set (i.e., mean, median, mode) is determined by the nature of the data and the question to be answered.
- Outliers affect the mean, median, and mode in different ways.
- Data interpretation is enhanced by numerical measures telling how data are distributed.

BIG IDEA #21

CHANCE: The chance of an event occurring can be described numerically by a number between 0 and 1 inclusive and used to make predictions about other events.

Examples of Mathematical Understandings:

- Probability can provide a basis for making predictions.
- Some probabilities can only be determined through experimental trials.
- An event that is certain to happen will always happen (The probability is 1.) and an event that is impossible will never happen (The probability is 0.).

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